Available at
www.ElsevierComputerScience.com
POWERED BY SCIENCE DIRECT
Computer Aided Geometric Design 20 (2003) 469–492
COMPUTER AIDED GEOMETRIC DESIGN
www.elsevier.com/locate/cagd

C^1 Hermite interpolation using MPH quartic

Gwang-Il Kim [a,*], Min-Ho Ahn [b,1]

[a] *Mathematics Major and RINS, Division of Mathematics and Information Statistics, College of Natural Sciences, Gyeong-Sang National University, Gyeong-Sang, South Korea*
[b] *Department of Mathematics, Pohang University of Science and Technology, Pohang, South Korea*

Received 14 May 2002; received in revised form 11 June 2003; accepted 17 June 2003

Abstract

In this paper, we study the C^1 Hermite interpolation problem using Minkowski Pythagorean Hodograph (MPH) quartics in $\mathbb{R}^{2,1}$. As a preliminary step, we characterize MPH curves in $\mathbb{R}^{2,1}$ by the roots of the hodographs of their complexified spine curves. We present two schemes for this interpolation problem: one is a subdivision scheme using direct C^1 interpolation and the other is a two step scheme using a new concept, $C^{1/2}$ interpolation.

Keywords: Pythagorean hodograph; Minkowski Pythagorean hodograph; C^1 Hermite interpolation; Subdivision scheme; $C^{1/2}$ Hermite interpolation

1. Introduction

Pythagorean hodograph curves introduced by Farouki (Farouki and Sakkalis, 1990) have their roots in the rational parameterization of curves and surfaces in the practical field of computer aided geometric design. After Farouki's introduction of Pythagorean Hodograph Curves, there has been vast researches on this class of curves by himself and others (Farouki, 1992, 1994, 1996, 1997; Farouki et al., 1998; Walton and Meek, 1996) and some related ones on a special class of curves called Minkowski Pythagorean hodograph curves (MPH curves) (Moon, 1999; Choi et al., 1999, 2002; Choi and Lee, 2000).

MPH curves introduced by Moon (1999) also have their roots in the rational parametrization of curves and surfaces. For example, as Hilgarter et al. (1999) pointed out, the offset with varying distance function

* Corresponding author.
E-mail addresses: gikim@nongae.gsnu.ac.kr (G.-I. Kim), mhahn@digitalaria.com (M.-H. Ahn).
[1] Present address: Visual Information Processing Lab. Digital Aria Co., Ltd., South Korea.

0167-8396/$ – see front matter
doi:10.1016/j.cagd.2003.06.001

$r(t)$ given by the spine curve $m(t)$ in $\mathbb{R}^2$ admits a rational parameterization over $\mathbb{R}$ in accordance with $m(t)$ if and only if $(m(t), r(t))$ is a MPH curve in $\mathbb{R}^{2,1}$. Of course, if the radius function $r(t)$ is constant, then the offset reduced to the classical offset and MPH condition reduce to PH condition. For canal surface which is the envelope of one parameter family of moving spheres, we know that the canal surface given by the spine curve m and the radius function r admits a rational parameterization over the reals in accordance with the spine curve m if and only if (m, r) is a space-like curve (Hilgarter et al., 1999; Peternell and Pottmann, 1997) which is a natural model of the one parameter family of spheres of varying radius in $\mathbb{R}^n$ that its spine curve is m and its radius function is $r(t)$ in the Minkowski space $\mathbb{R}^{n,1}$. However, the rational parametrization of canal surfaces given by a space-like curve (m, r) requires the factorization procedure in the rational parametrization algorithm (Hilgarter et al., 1999; Peternell and Pottmann, 1997) spending the computational cost. On the other hand, if we restrict the spine curve of canal surface to the MPH curve, then factorization procedure is not required.

The medial axis transform, in symbols $MAT(*)$, closely related to swept volume of spheres is defined to be the set of pairs consisting of centers and radii of the spheres, in other words the image of the cyclographical mapping of spheres, maximally inscribed in the domain. MPH curves are also used to compute the medial axis transform of a domain (Choi et al., 1999). Consider an one parameter family $\mathcal{C} = \{B(m(t), r(t)) \mid t \in I\}$ of spheres of radius function $r(t)$ with spine curve $m(t)$. The swept volume of $\mathcal{C}$, in symbols $SV(\mathcal{C})$, is determined by the envelope of spheres $\bigcup\{B(m(t), r(t)) \mid t \in I\}$. If the curve (m, r) embedded in the Minkowski space $\mathbb{R}^{n,1}$ is a MPH curve, the boundary of the swept volume can be rationally parametrized more efficiently as stated in offset cases. Especially in $\mathbb{R}^{2,1}$, for $\Omega = SV(\mathcal{C})$ we have the following results which explain the interesting relation between the swept volume and the medial axis transform;

$$MAT(\Omega) \neq \mathcal{C},$$
$$SV\big(MAT\big(SV(\mathcal{C})\big)\big) = SV(\mathcal{C}),$$

where

$$MAT(\Omega) = \big\{(m, r) \mid \operatorname{dist}(p, m) = r = \operatorname{dist}(q, m), \text{ for some } p, q \in \partial\Omega \text{ s.t. } p \neq q\big\}$$

if (m, r) is a MPH curve.

In this paper, we focus on MPH curves in the Minkowski space $\mathbb{R}^{2,1}$. In the first part of this paper, we characterize MPH curves by the roots of the hodographs of them using the complex representation of plane curves introduced by Farouki (1994). By this characterization, we find regular MPH curves can have even degree even though regular PH curves admit only odd degree. This means that for Hermite interpolation, in case of MPH curves, the required degree for MPH curve is less than that of PH curve. For example, for C^1 interpolation, we do not need MPH quintic curves as expected in the PH context. *MPH quartic curves* are enough. In the second part of this paper, we solve C^1 Hermite interpolation problem with MPH quartic curves. For graphical applications, G^1 interpolation is sufficient. However, for tool path generation, we need C^1 (or higher) interpolation. We know that for C^1 interpolation, the cubic polynomials are sufficient in general case and the PH quintic curves are needed in the plane PH curve context (Farouki and Neff, 1995; Moon et al., 2001). In the space PH context, PH cubic curves were used in G^1 interpolation problem in (Jüttler and Mäuer, 1999). Recently Farouki et al. solved C^1 Hermite interpolation problem with helical PH quintic space curves and spatial C^2 Hermite interpolation problem with PH quintic curves in (Farouki et al., 2003b) and (Farouki et al., 2003a), respectively. In the MPH context, G^1 interpolation using the MPH cubic curves is studied in (Choi et al., 1999). However,

C^1 interpolation is not achieved until now. In this paper, we achieve it with MPH quartic curves using $C^{1/2}$ *interpolation* explained in Section 4.

2. Roots characterization of MPH curves

Recently, Moon (1999) showed that the necessary and sufficient condition for a polynomial curve $\gamma(t) = (a(t), b(t), c(t))$ in the three-dimensional Minkowski space $\mathbb{R}^{2,1}$ to be PH is that for its velocity function $\sigma(t)$ (,that is, $\sigma(t)^2 = a'(t)^2 + b'(t)^2 - c'(t)^2$) there exist four polynomial functions $u(t)$, $v(t)$, $\rho(t)$, $\omega(t)$ satisfying the following relations:

$$\begin{aligned}
\sigma(t) &= u(t)^2 + v(t)^2 - \omega(t)^2 - \rho(t)^2,\\
a'(t) &= u(t)^2 - v(t)^2 + \omega(t)^2 - \rho(t)^2,\\
b'(t) &= 2u(t)v(t) - 2\rho(t)\omega(t),\\
c'(t) &= 2u(t)\omega(t) - 2\rho(t)v(t).
\end{aligned}$$

In this section, we reformulate this theorem into more convenient and useful form using complex representation. We will seek the results which can be obtained by observing the roots of polynomials in complex representation as in the characterization of PH curves by their roots (Ahn and Kim, t.a.).

Definition 1. Two complex numbers z_1,z_2 are *semiequal* if $z_1 = z_2$ or $z_1 = \overline{z_2}$, (denote by $z_1 \approx z_2$) and z_1, z_2 are distinct up to conjugate if z_1, z_2 are not semiequal.

Definition 2. For a real polynomial $h(t)$ such that $h(t) \geqslant 0$ for all t, denote $[\mathbf{h}(\mathbf{t})]$ the set of all polynomial curves $\alpha(t)$ such that $a(t)^2 + b(t)^2 = h(t)$. We say that two polynomial curves $\alpha_1(t) = (a(t), b(t))$ and $\beta(t) = (c(t), d(t))$ are of the same class $[h(t)]$ if $a(t)^2 + b(t)^2 = c(t)^2 + d(t)^2 = h(t)$. A polynomial $f(t)$ is a member of $[h(t)]$ if there exists a polynomial $g(t)$ such that $\beta(t) = (f(t), g(t))$ or $\beta(t) = (g(t), f(t))$ is a member of $[h(t)]$.

Theorem 3. *A polynomial curve $\alpha(t) = (a(t), b(t), c(t))$ in the three-dimensional Minkowski space $\mathbb{R}^{2,1}$ is a MPH curve if and only if $c'(t)$ is a member of $[a'(t)^2 + b'(t)^2]$.*

Proof. Suppose $\alpha(t)$ is a MPH curve, that is, its velocity function $\sigma(t) = (a'(t)^2 + b'(t)^2 - c'(t)^2)^{1/2}$ is a member of $\mathcal{P}[t]$. Thus, $a'(t)^2 + b'(t)^2 = c'(t)^2 + \sigma(t)^2$. That is, $c'(t)$ is a member of the class $[a'(t)^2 + b'(t)^2]$.

Conversely, if $c'(t)$ is a member of the class $[a'(t)^2 + b'(t)^2]$, then there exists a polynomial $k(t)$ such that $a'(t)^2 + b'(t)^2 = c'(t)^2 + k(t)^2$. This polynomial $k(t)$ is a velocity function of a polynomial curve $\alpha(t)$ in the three-dimensional Minkowski $\mathbb{R}^{2,1}$, and hence $\alpha(t)$ is a MPH curve. □

The above theorem has an important meaning: *Given a nonnegative polynomial $h(t)$, we can determine MPH curves in $\mathbb{R}^{2,1}$ whose velocity functions are members of the equivalence class* $[h(t)]$. To make this fact more clear, we need the relation between the members of the class $[h(t)]$.

Lemma 4. *Suppose two plane polynomial curves $\alpha_1(t) = (a_1(t), b_1(t))$ and $\alpha_2(t) = (a_2(t), b_2(t))$ are of the same class $[h(t)]$. Assume the complexified curves $\beta_j(t) = a_j(t) + ib_j(t)$ for $j = 1, 2$ are factorized over the field of complex numbers $\mathbb{C}$. Then each root of $\beta_1(t)$ is semi-equal to one of the roots of $\beta_2(t)$.*

Proof. By the assumption, $\|\beta_1(t)\|^2 = \|\beta_2(t)\|^2 = h(t)$. Let r_i for $1 \leqslant i \leqslant k_1$ be the real zeros of $h(t)$ and $c_j, \overline{c_j}$ for $1 \leqslant j \leqslant k_2$ the complex (not real) zeros of $h(t)$ and their conjugates. Clearly, r_is are zeros of both β_1 and β_2 and all complex (not real) zeros of β_1 and β_2 must be c_j or $\overline{c_j}$ for all j. That is, each root of $\beta_1(t)$ is semi-equal to one of the roots of $\beta_2(t)$. □

Theorem 3 and Lemma 4 tell us that if a polynomial curve $\alpha(t) = (a(t), b(t), c(t))$ in the three-dimensional Minkowski space $\mathbb{R}^{2,1}$ is a MPH curve, then α is controlled by the plane polynomial curve $\tilde{\alpha}(t) = (a(t), b(t))$ and the possible forms of α are completely determined by the linear factors of complexified hodograph of $\tilde{\alpha}$. In summary, we get the following theorem.

Theorem 5. *Suppose $\alpha(t) = (a(t), b(t), c(t))$ is a polynomial curve in the three-dimensional Minkowski space $\mathbb{R}^{2,1}$. Let $t - r_i$ $(1 \leqslant i \leqslant k_1)$ and $t - c_j$ $(1 \leqslant j \leqslant k_2)$ be the linear factors with real coefficients and with complex coefficients of a complexified plane polynomial curve $\gamma(t) = a'(t) + ib'(t)$. (That is, $\gamma(t) = \mathbf{k}\prod_{i=1}^{k_1}(t - r_i)\prod_{j=1}^{k_2}(t - c_j)$.) Let d_j be a complex number semi-equal to c_j for $j = 1, \ldots, k_2$.*

$\alpha(t)$ is a MPH curve if and only if $c'(t)$ is the real or imaginary part of a new complexified plane polynomial curve $\delta(t) = \tilde{\mathbf{k}}\prod_{i=1}^{k_1}(t - r_i)\prod_{j=1}^{k_2}(t - d_j)$, where $\tilde{\mathbf{k}}$ is a complex number such that $\|\mathbf{k}\| = \|\tilde{\mathbf{k}}\|$.

Throughout this paper, the complexified curve $\delta(t)$ represented in Theorem 5 is called the dual curve of $\gamma(t)$.

Example 6. Given two polynomials $a(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 5t + 1$ and $b(t) = t^2 - t + 2$, we will determine all possible MPH curves $\alpha(t) = (a(t), b(t), c(t))$. By factorizing the complexified plane polynomial curve $\gamma(t) = a'(t) + ib'(t)$ over $\mathbb{C}$, we get $\gamma(t) = (t - 1 - i)(t - 2 + 3i)$. Since $\mathbf{k} = 1$, $\tilde{\mathbf{k}} = \mathrm{e}^{i\theta}$. By Theorem 5, $c'(t)$ must be the real part or imaginary part of the following polynomials:

$$\mathrm{e}^{i\theta}(t - 1 + i)(t - 2 + 3i), \qquad \mathrm{e}^{i\theta}(t - 1 - i)(t - 2 - 3i), \qquad \mathrm{e}^{i\theta}(t - 1 + i)(t - 2 - 3i).$$

Especially when $\theta = 0$, $c'(t)$ is one of $t^2 - 3t - 1$, $4t - 5$, $-4t + 5$, $t^2 - 3t + 5$, $-2t + 1$. Thus, all possible choices of $\alpha(t) = (a(t), b(t), c(t))$ are determined by the choice of $c(t)$:

$$c(t) = \tfrac{1}{3}t^3 - \tfrac{3}{2}t^2 - t + c_0, \quad 2t^2 - 5t + c_0, \quad -2t^2 + 5t + c_0,$$
$$-t^2 + t - c_0, \quad \text{or} \quad \tfrac{1}{3}t^3 - \tfrac{3}{2}t^2 + 5t + c_0.$$

The next example shows how to determine MPH curves whose velocity function are members of pre-assigned class $[h(t)]$.

Example 7. Let $h(t) = t^4 - 6t^3 + 23t^2 - 34t + 26$. By factorizing $h(t)$ over $\mathbb{C}$, we get

$$h(t) = (t - 1 - i)(t - 1 + i)(t - 2 - 3i)(t - 2 + 3i).$$

Thus the possible choices for $a'(t)$ and $b'(t)$ are

$$\begin{aligned}\big(a'(t), b'(t)\big) = \big(\big(t^2 - 3t + 5\big), (2t - 1)\big), &\quad \big(\big(t^2 - 3t - 1\big), (-4t + 5)\big),\\ \big(\big(t^2 - 3t - 1\big), (4t - 5)\big), &\quad \big(\big(t^2 - 3t + 5\big), (-2t + 1)\big).\end{aligned}$$

Since $\mathbf{k} = 1$, let $\tilde{\mathbf{k}} = \cos\theta + i\sin\theta$ for real number θ. The corresponding choices for $c'(t)$ are $\cos\theta(t^2 - 3t - 1) + \sin\theta(4t - 5)$ and $\cos\theta(t^2 - 3t + 5) + \sin\theta(-2t + 1)$.

Theorem 5 means that the polynomial MPH curve in $\mathbb{R}^{2,1}$ is completely characterized by the roots of the hodograph of its spine curve. In the following subsections, computing the speed function and the curvature of the envelope which a MPH curve generates, we consider how the spine curve and its dual curve whose hodograph has roots semi-equal to those of the spine curve work on the resulting envelope of the MPH curve.

2.1. Envelope

First, consider a polynomial MPH curve $\gamma(t) = (x(t), y(t), r(t)) \in \mathbb{R}^{2,1}$. Then the equation of the resulting envelope of the family of circles centered at $(x(t), y(t))$ with radius $r(t)$, denoted by $E_\pm(t)$, is given as follows:

$$E_\pm(t) = \alpha(t) - r(t)m_\pm(t),$$

where

$$\begin{aligned}\alpha(t) &= \big(x(t), y(t)\big),\\ \sigma(t) &= \sqrt{x'(t)^2 + y'(t)^2 - r'(t)^2},\\ m_\pm(t) &= \left(\frac{r'(t)x'(t) \pm y'(t)\sigma(t)}{x'(t)^2 + y'(t)^2}, \frac{r'(t)y'(t) \mp x'(t)\sigma(t)}{x'(t)^2 + y'(t)^2}\right).\end{aligned}$$

Using complexified curves $\alpha(t) = x(t) + iy(t)$, $\alpha'(t) = x'(t) + iy'(t)$ and $v(t) = r'(t) + i\sigma(t)$, we can rewrite the envelope as follows:

$$E_\pm(t) = \alpha(t) - r(t)\omega_\pm(t), \tag{2.1}$$

where

$$\omega_+(t) = \frac{\alpha'(t)\overline{v(t)}}{x'(t)^2 + y'(t)^2}, \qquad \omega_-(t) = \frac{\alpha'(t)v(t)}{x'(t)^2 + y'(t)^2}.$$

Remark 8. Since $\gamma(t)$ is a MPH curve, α' and v are of the same class (i.e., $\|\alpha'\| = \|v\|$). So, we have that $\|\omega_\pm(t)\| = 1$.

Next, we compute the speed function $\mathrm{Sp}(t)$ of the envelope curve with the help of the following lemma:

Proposition 9.

(1) *For a complexified plane curve $\alpha(t)$, the following two equations hold*:

$$\left(\frac{\alpha'}{\|\alpha'\|}\right)' = \kappa_\alpha i\alpha', \qquad \kappa_{\bar{\alpha}} = -\kappa_\alpha.$$

(2) *For a complexified plane curve* $\alpha(t)$, $\beta(t)$ *with* $\|\alpha'\| = \|\beta'\|$, *the derivative of* $\omega(t) = \frac{\alpha'}{\|\alpha'\|}\frac{\beta'}{\|\beta'\|}$ *is given as follows.*

$$\omega' = i\omega\|\alpha'\|(\kappa_\alpha + \kappa_\beta)$$

where κ_α *and* κ_β *are curvature of* α *and* β*, respectively.*

Proof. (1) From the Frenet formula, we get

$$\left(\frac{\alpha'}{\|\alpha'\|}\right)' = T'(t) = \kappa_\alpha\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)N(t) = \kappa_\alpha\|\alpha'\|i\frac{\alpha'}{\|\alpha'\|} = \kappa_\alpha i\alpha'.$$

Note that $\mathrm{Im}(\|\alpha'\|^2) = \mathrm{Im}(\alpha'\bar{\alpha}') = 0$ and $\kappa_\alpha = \frac{\mathrm{Im}(\overline{\alpha'}\alpha'')}{\|\alpha'\|^3}$. Thus

$$\kappa_{\bar{\alpha}} = \frac{\mathrm{Im}(\overline{\bar{\alpha}'}\bar{\alpha}'')}{\|\bar{\alpha}'\|^3} = \frac{\mathrm{Im}(\alpha'\overline{\alpha''})}{\|\alpha'\|^3} = -\frac{\mathrm{Im}(\overline{\alpha'}\alpha'')}{\|\alpha'\|^3} = -\kappa_\alpha.$$

(2) ω' is computed as follows.

$$\omega' = \left(\frac{\alpha'}{\|\alpha'\|}\frac{\beta'}{\|\beta'\|}\right)' = \left(\frac{\alpha'}{\|\alpha'\|}\right)'\left(\frac{\beta'}{\|\beta'\|}\right) + \left(\frac{\alpha'}{\|\alpha'\|}\right)\left(\frac{\beta'}{\|\beta'\|}\right)'$$
$$= \kappa_\alpha i\alpha'\left(\frac{\beta'}{\|\beta'\|}\right) + \frac{\alpha'}{\|\alpha'\|}\kappa_\beta i\beta' = \frac{i\alpha'\beta'}{\|\alpha'\|}(\kappa_\alpha + \kappa_\beta) = i\omega\|\alpha'\|(\kappa_\alpha + \kappa_\beta). \qquad \square$$

Now, we compute $\mathrm{Sp}(t)$. Note that $E'_\pm(t) = \alpha'(t) - r'(t)\omega_\pm(t) - r(t)\omega'_\pm(t)$. So, $\mathrm{Sp}(t)^2$ is given as follows:

$$\begin{aligned}\mathrm{Sp}(t)^2 &= E'_\pm(t)\overline{E'_\pm(t)}\\ &= \big(\alpha'(t) - r'(t)\omega_\pm(t) - r(t)\omega'_\pm(t)\big)\overline{\big(\alpha'(t) - r'(t)\omega_\pm(t) - r(t)\omega'_\pm(t)\big)}\\ &= \big\|\alpha'(t)\big\|^2 + (r')^2\|\omega_\pm\|^2 + r^2\|\omega'_\pm\|^2 - r'(\alpha'\overline{\omega_\pm} + \overline{\alpha}'\omega_\pm)\\ &\quad - r(\alpha'\overline{\omega'_\pm} + \overline{\alpha}'\omega'_\pm) + rr'(\overline{\omega_\pm}\omega'_\pm + \omega_\pm\overline{\omega'_\pm}).\end{aligned}$$

By Remark 8, we get $\overline{\omega_\pm}\omega'_\pm + \omega_\pm\overline{\omega'_\pm} = 0$. Henceforth

$$\mathrm{Sp}(t)^2 = \big\|\alpha'(t)\big\|^2 + (r')^2 + r^2\|\omega'_\pm\|^2 - 2r'\,\mathrm{Re}(\alpha'\overline{\omega_\pm}) - 2r\,\mathrm{Re}(\alpha'\overline{\omega'_\pm}). \tag{2.2}$$

Moreover, for

$$\begin{aligned}\alpha'\overline{\omega_\pm} &= \alpha'\overline{\alpha'}v(t)/\big(x'^2 + y'^2\big) \quad (\text{or } \alpha'\overline{\alpha'}\,\overline{v(t)}/(x'^2 + y'^2))\\ &= v(t) \quad (\text{or } \overline{v(t)}),\end{aligned}$$

we have

$$\mathrm{Re}(\alpha'\overline{\omega_\pm}) = r'. \tag{2.3}$$

Let $V(t) = \int_0^t v(s)\,\mathrm{d}s$. Then by Proposition 9, $(\omega_\pm)'$ are given as follows:

$$\begin{aligned}(\omega_+)' &= i\omega_+\|\alpha'\|(\kappa_\alpha + \kappa_{\overline{V}}) = i\omega_+\|\alpha'\|(\kappa_\alpha - \kappa_V),\\ (\omega_-)' &= i\omega_-\|\alpha'\|(\kappa_\alpha + \kappa_V).\end{aligned}$$

Thus we have

$$\begin{aligned}\mathrm{Re}(\alpha'\overline{\omega'_\pm}) &= \mathrm{Re}\big(\alpha'(-i)\overline{\omega_\mp}\|\alpha'\|(\kappa_\alpha \mp \kappa_V)\big)\\ &= \mathrm{Im}(\alpha'\overline{\omega_\pm})\|\alpha'\|(\kappa_\alpha \mp \kappa_V)\\ &= \pm\sigma\|\alpha'\|(\kappa_\alpha \mp \kappa_V).\end{aligned} \tag{2.4}$$

Plugging (2.3), (2.4) into (2.2), we get the following

$$\begin{aligned}\mathrm{Sp}(t)^2 &= \|\alpha'\|^2 + r'^2 + r^2\|\omega_\pm\|^2\|\alpha'\|^2(\kappa_\alpha \mp \kappa_V)^2 - 2r'^2 - 2r\big(\pm\sigma\|\alpha'\|(\kappa_\alpha \mp \kappa_V)\big)\\ &= \big(\|\alpha'\|^2 - r'^2\big) \mp 2r\sigma\|\alpha'\|(\kappa_\alpha \mp \kappa_V) + r^2\|\alpha'\|^2(\kappa_\alpha \mp \kappa_V)^2\\ &= \big(r\|\alpha'\|(\kappa_\alpha \mp \kappa_V) \mp \sigma\big)^2.\end{aligned}$$

Remark 10. (1) If we apply the fact that the conjugation for complex curves is equivalent to the reflection for plane curves, then the second part of (1) in Proposition 9 is clear.

(2) Since $\|\alpha'\|\kappa_\alpha = \frac{\mathrm{Im}(\overline{\alpha'}\alpha'')}{\|\alpha'\|^2}$, the above equation implies that for a polynomial MPH curve $\gamma(t) = (x(t), y(t), r(t))$, the derived envelopes are (plane) PH curves whose speeds are rational.

2.2. *Curvature of envelope*

In this subsection, we compute the curvature of the envelope. Let $\mathrm{Sp}(t), T(t), N(t)$ and $\kappa(t)$ be the speed function, the unit tangent vector field, the unit normal vector field and the curvature of envelope curve, respectively. Since we are dealing with sweeping of circle, it is clear that

$$\alpha(t) = E_\pm(t) \mp r(t)N(t). \tag{2.5}$$

Thus we get $\omega_\pm(t) = \mp N(t)$ and

$$\omega'_\pm(t) = \mp N'(t) = \pm\kappa(t)\,\mathrm{Sp}(t)T(t).$$

So,

$$\kappa = \pm\frac{\omega'_\pm\overline{T}}{\mathrm{Sp}} = \pm\frac{i\omega_\pm\|\alpha'\|(\kappa_\alpha \mp \kappa_V)\overline{E'_\pm}}{\mathrm{Sp}^2} = \pm\frac{i\omega_\pm\|\alpha'\|(\kappa_\alpha \mp \kappa_V)(\overline{\alpha'} - r'\overline{\omega_\pm} - r\overline{\omega'_\pm})}{\mathrm{Sp}^2}.$$

Using the followings:

(1) $\omega_\pm\overline{\alpha'} = \frac{\alpha'\bar{v}\overline{\alpha'}}{x'^2+y'^2}$ (or $\frac{\alpha'\bar{v}\overline{\alpha'}}{x'^2+y'^2}$) $= \bar{v}$ (or v) $= r' \mp i\sigma$,
(2) $\omega_\pm\overline{\omega_\pm} = 1$,
(3) $\omega_\pm\overline{\omega'_\pm} = -i\|\alpha'\|(\kappa_\alpha \mp \kappa_V)$,

finally, for $A_\pm = \|\alpha'\|(\kappa_\alpha \pm \kappa_V)$, we have

$$\kappa = \pm\frac{A_\mp(\pm\sigma + ir') - iA_\mp r' - rA_\mp^2}{\mathrm{Sp}^2} = \mp\frac{A_\mp}{(rA_\mp \mp \sigma)}.$$

3. Hermite interpolation

We want to find a regular PH quartic in a Minkowski space $\mathbb{R}^{2,1}$ which satisfying the following a first order Hermite data:

$$\alpha(0) = \mathbf{P}_0, \qquad \alpha(1) = \mathbf{P}_1, \tag{3.1}$$

$$\alpha'(0) = \mathbf{D}_0, \qquad \alpha'(1) = \mathbf{D}_1, \tag{3.2}$$

where $\mathbf{P}_k = (x_k, y_k, r_k)$ and $\mathbf{D}_k = (d_k^x, d_k^y, e_k)$. We will denote $\mathbf{z}_k = x_k + iy_k$ and $\mathbf{d}_k = d_k^x + id_k^y$.

In fact, we are seeking four polynomials $x(t), y(t), r(t), \sigma(t)$ such that $(x'(t))^2 + (y'(t))^2 = (r'(t))^2 + (\sigma'(t))^2$. Let $\tilde{\alpha}(t) = x(t) + iy(t)$ and $\beta(t) = r(t) + i\sigma(t)$. We know that if $\tilde{\alpha}'(t)$ is factorized into $\mathbf{k}(t-\omega_1)(t-\omega_2)(t-\omega_3)$, then $\beta'(t)$ is given by $\mathbf{k}\mathrm{e}^{i\theta}(t-\omega_1^*)(t-\omega_2^*)(t-\omega_3^*)$ where ω_j^* is a complex number semi-equal to ω_j $j = 1, 2, 3$. From the Hermite data, we get the following constraints:

$$\mathbf{z}_1 = \mathbf{k}\left(\tfrac{1}{4} - \tfrac{1}{3}S_1 + \tfrac{1}{2}S_2 - S_3\right), \tag{3.3}$$

$$\mathbf{d}_0 = -\mathbf{k}S_3, \tag{3.4}$$

$$\mathbf{d}_1 = \mathbf{k}(1 - S_1 + S_2 - S_3), \tag{3.5}$$

$$\beta_1 = \mathrm{e}^{i\theta}\mathbf{k}\left(\tfrac{1}{4} - \tfrac{1}{3}S_1^* + \tfrac{1}{2}S_2^* - S_3^*\right), \tag{3.6}$$

$$\tilde{\mathbf{d}}_0 = -\mathrm{e}^{i\theta}\mathbf{k}S_3^*, \tag{3.7}$$

$$\tilde{\mathbf{d}}_1 = \mathrm{e}^{i\theta}\mathbf{k}\left(1 - S_1^* + S_2^* - S_3^*\right), \tag{3.8}$$

where $\tilde{\mathbf{d}}_j = e_j \pm i\sqrt{\|\mathbf{d}_j\|^2 - e_j^2}$ for $j = 1, 2$ and S_j and S_j^* are the jth symmetric polynomials over $\{\omega_j\}$ and $\{\omega_j^*\}$, respectively.

Depending on the choices of ω_j^*, we have 4 cases.

(I) $\omega_j^* = \omega_j$ for all $j = 1, 2, 3$.
(II) $\omega_1^* = \overline{\omega}_1$ and $\omega_j^* = \omega_j$ for $j = 2, 3$.
(III) $\omega_j^* = \overline{\omega}_j$ for $j = 1, 2$ and $\omega_3^* = \omega_3$.
(IV) $\omega_j^* = \overline{\omega}_j$ for $j = 1, 2, 3$.

Case I. In this case, $S_j^* = S_j$ for $j = 1, 2, 3$. From Eqs. (3.4), (3.5), (3.7), and (3.8), we get

$$\frac{\mathbf{d}_0}{\tilde{\mathbf{d}}_0} = \mathrm{e}^{-i\theta} = \frac{\mathbf{d}_1}{\tilde{\mathbf{d}}_1}.$$

θ is computed from the above equation and the system becomes underdetermined. If

$$\frac{\mathbf{d}_0}{\tilde{\mathbf{d}}_0} = \frac{\mathbf{d}_1}{\tilde{\mathbf{d}}_1}, \qquad r_1 = \mathrm{Re}\left(\mathrm{e}^{i\theta}\mathbf{z}_1\right) \tag{3.9}$$

holds, then there exists infinitely many solutions for $\mathbf{k}, \omega_j$. If not, then there exist no solutions. Note that if Eq. (3.9) holds, then $\beta(t) = \mathrm{e}^{i\theta}\tilde{\alpha}(t)$, that is, the curve $\beta(t)$ is a rotated curve of $\tilde{\alpha}(t)$ which is a projected curve of $\alpha(t)$. It is a degenerate case. To solve this Hermite interpolation problem, it suffice to use a cubic curve. Strategy is very simple: First solve the C^1 Hermite interpolation problem in $\mathbb{R}^2$ with projected data using planar cubic $\tilde{\alpha}(t)$. Then $\beta(t)$ is a rotated curve of $\tilde{\alpha}(t)$ by θ. Finally, taking the real part of $\beta(t)$ gives the solution. If we need quartic curve $\alpha(t)$, degree elevation can be performed.

Case II. From Eqs. (3.4), (3.5), (3.7), and (3.8), we get

$$\frac{\mathbf{d}_0}{\tilde{\mathbf{d}}_0} = \mathrm{e}^{-i\theta}\frac{\omega_1}{\overline{\omega}_1},$$
$$\frac{\mathbf{d}_1}{\tilde{\mathbf{d}}_1} = \mathrm{e}^{-i\theta}\frac{(1-\omega_1)}{(1-\overline{\omega}_1)}.$$

Dividing above two equations, we get

$$\frac{\mathbf{a}_1}{\mathbf{a}_0} = \frac{\bar{\lambda}}{\lambda},$$

where $\mathbf{a}_j = \mathbf{d}_j/\tilde{\mathbf{d}}_j$ for $j = 1, 2$ and $\lambda = \frac{1-\omega_1}{\omega_1}$. Note that λ is written as

$$\lambda = r\mathrm{e}^{i\eta}, \qquad \eta = \frac{\theta_1 - \theta_0}{2}, \tag{3.10}$$

where $\theta_j = \arg(\mathbf{a}_j)$ for $j = 1, 2$ (note that r can have negative values).

Thus,

$$\omega_1 = \frac{1}{1 + r\mathrm{e}^{i\eta}}.$$

Once we find ω_1, it is quite easy to find ω_2 and ω_3.

From Eqs. (3.3)–(3.5), we get

$$\omega_1(\omega_2\omega_3)\mathbf{z}_1 = -\mathbf{d}_0\Big(\tfrac{1}{4} - \tfrac{1}{3}(\omega_1+\omega_2+\omega_3) + \tfrac{1}{2}\big(\omega_1(\omega_2+\omega_3)+\omega_2\omega_3\big) - \omega_1(\omega_2\omega_3)\Big),$$
$$\omega_1(\omega_2\omega_3)\mathbf{d}_1 = -\mathbf{d}_0\big(1 - (\omega_1+\omega_2+\omega_3) + \omega_1(\omega_2+\omega_3) + \omega_2\omega_3 - \omega_1(\omega_2\omega_3)\big).$$

Let T_1 and T_2 be the symmetric polynomials over ω_2 and ω_3. Then we get the following linear equation for T_1, T_2:

$$\begin{pmatrix} \omega_1\mathbf{z}_1 + \mathbf{d}_0(\frac{1}{2} - \omega_1) & \mathbf{d}_0(-\frac{1}{3} + \frac{1}{2}\omega_1) \\ \omega_1\mathbf{d}_1 + \mathbf{d}_0(1-\omega_1) & -\mathbf{d}_0(1-\omega_1) \end{pmatrix} \begin{pmatrix} T_2 \\ T_1 \end{pmatrix} = \begin{pmatrix} \mathbf{d}_0(-\frac{1}{4} + \frac{1}{3}\omega_1) \\ -\mathbf{d}_0(1-\omega_1) \end{pmatrix}.$$

Finally, $\mathbf{k}$ is given by $\mathbf{k} = -\frac{\mathbf{d}_0}{\omega_1 T_2}$. Plugging theses quantities into (3.6), we get

$$\begin{aligned} r_1 = \mathrm{Re}(\beta_1) &= \mathrm{Re}\big(\mathrm{e}^{i\theta}\mathbf{k}\big(\tfrac{1}{4} - \tfrac{1}{3}S_1^* + \tfrac{1}{2}S_2^* - S_3^*\big)\big) \\ &= \mathrm{Re}\bigg(-\frac{\tilde{\mathbf{d}}_0}{\overline{\omega}_1 T_2}\big(\tfrac{1}{4} - \tfrac{1}{3}(\overline{\omega}_1 + T_1) + \tfrac{1}{2}(\overline{\omega}_1 T_1 + T_2) - \overline{\omega}_1 T_2\big)\bigg). \end{aligned} \tag{3.11}$$

Thus the only known in the above equation is r. After tedious algebraic manipulation using $\|\tilde{\mathbf{d}}_j\|^2 = \|\mathbf{d}_j\|^2$, we get the following quadratic equation in r:

$$c_2r^2 + c_1r + c_0 = 0, \tag{3.12}$$

where c_0, c_1, c_2 are given as follows:

$$2c_0 = -6r_1 + \mathrm{Re}\big(\mathbf{a}_0^{-1}\big(6\mathbf{z}_1 + \mathbf{d}_1\mathrm{e}^{-2i\eta} - \mathbf{d}_1\big)\big),$$
$$2c_1 = 12r_1\cos(\eta) + \mathrm{Re}\big(\mathbf{a}_0^{-1}\big(\mathbf{d}_1\mathrm{e}^{-i\eta} - 12\mathbf{z}_1\mathrm{e}^{-i\eta} - \mathbf{d}_1\mathrm{e}^{-3i\eta}\big)\big) + 2\,\mathrm{Im}(\tilde{\mathbf{d}}_0)\sin(\eta),$$
$$2c_2 = -6r_1 + \mathrm{Re}\big(\mathbf{a}_0^{-1}\big(\mathbf{d}_0 - \mathbf{d}_0\mathrm{e}^{-2i\eta} + 6\mathbf{z}_1\mathrm{e}^{-2i\eta}\big)\big).$$

See Appendix A for derivation.

Case III. Note that $\mathrm{Re}(\beta(t)) = \mathrm{Re}(\overline{\beta(t)})$. Hence

$$r_1 = \mathrm{Re}\big(\beta(1)\big) = \mathrm{Re}\big(\overline{\beta(1)}\big),$$
$$e_0 = \mathrm{Re}(\tilde{\mathbf{d}}_0) = \mathrm{Re}\big(\beta'(0)\big) = \mathrm{Re}\big(\overline{\beta'(0)}\big),$$
$$e_1 = \mathrm{Re}(\tilde{\mathbf{d}}_1) = \mathrm{Re}\big(\beta'(1)\big) = \mathrm{Re}\big(\overline{\beta'(1)}\big),$$

and

$$\mathrm{Re}\big(\overline{\mathbf{k}e^{i\theta}(t-\overline{\omega_1})(t-\overline{\omega_2})(t-\omega_3)}\big) = \mathrm{Re}\big(\bar{\mathbf{k}}e^{-i\theta}(t-\omega_1)(t-\omega_2)(t-\overline{\omega_3})\big)$$
$$= \mathrm{Re}\big(\mathbf{k}e^{i\theta^*}(t-\omega_1)(t-\omega_2)(t-\overline{\omega_3})\big),$$

where $\theta^* = -2\theta_k - \theta$ and $\theta_k = \arg(\mathbf{k})$. From the definition of $\tilde{\mathbf{d}}_j = e_j \pm \sqrt{\|\mathbf{d}_j\|^2 - e_j^2}$, we have four cases for $\tilde{\mathbf{d}}_0, \tilde{\mathbf{d}}_1$. Thus if we solve case II for all possible four cases, then we can cover case III.

Case IV. It is similar to case III. By solving case I, we can also solve case IV.

Remark 11. As in the case I, if the following conditions are hold, then there are no MPH curve to solve C^1 Hermite interpolation problem.

$$\frac{\mathbf{d}_0}{\tilde{\mathbf{d}}_0} = \frac{\tilde{\mathbf{d}}_1}{\mathbf{d}_1}, \qquad r_1 \neq \mathrm{Re}\left(\frac{\mathbf{d}_0}{\tilde{\mathbf{d}}_0}\mathbf{z}_1\right).$$

Collecting the above results, we get the following algorithm for C^1 Hermite interpolation using MPH quartic curve.

Algorithm (C^1 *Hermite interpolation using MPH quartic*)
Input: $P_0, P_1, D_0, D_1 \in \mathbb{R}^{2,1}$
Output: MPH quartic
($*$ D_0, D_1 must be space-like vectors $*$)

1. Compute $\mathbf{z}_0, \mathbf{z}_1, \mathbf{d}_0, \mathbf{d}_1, \tilde{\mathbf{d}}_0, \tilde{\mathbf{d}}_1$.
2. ($*$ We have four cases according to the sign of imaginary parts of $\tilde{\mathbf{d}}_0, \tilde{\mathbf{d}}_1$. $*$)
3. $\mathbf{z}_1 \leftarrow \mathbf{z}_1 - \mathbf{z}_0$
4. **for** each 4 cases
5. **do** $\mathbf{a}_0 \leftarrow \frac{\mathbf{d}_0}{\tilde{\mathbf{d}}_0}$, $\mathbf{a}_1 \leftarrow \frac{\mathbf{d}_1}{\tilde{\mathbf{d}}_1}$
6. **if** $\mathbf{a}_0 = \mathbf{a}_1$
7. **then if** $r_1 = \mathrm{Re}(\mathbf{z}_1/\mathbf{a}_0)$
8. **then** solve the plane cubic Hermite interpolation problem
9. **else return** "There are no solution."
10. **else** solve case II

Although we took η as $\frac{\theta_1-\theta_0}{2}$ in Eq. (3.10), $\frac{\theta_1-\theta_0}{2} + \pi$ is also possible. Thus if a solution r for Eq. (3.12) is negative, then we must interpret it by $-re^{i(\frac{\theta_1-\theta_0}{2}+\pi)}$ as in the polar coordinate in plane. By the Algorithm C^1 *Hermite interpolation using MPH quartic*, we get 8 MPH quartics in generic case (see Fig. 1).

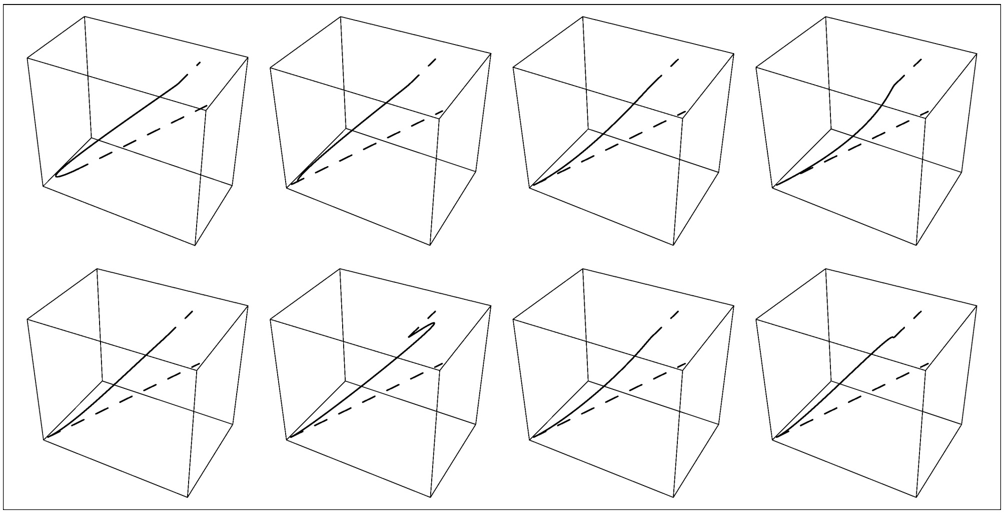

Fig. 1. 8 MPH quartics solving C^1 Hermite interpolation problem of an Hermite data $P_0 = (0,0,0)$, $D_0 = (0.9934, 0.07224, 0.8761)$; $P_1 = (0.6098, 0.4777, 0.0.6977)$, $D_1 = (0.0659, 0.2298, 0.0780)$. Dotted lines are given tangent vectors and solid curves are interpolating MPH quartics.

Remark 12. To guarantee the existence of solution, we need the discriminant of the quadratic Eq. (3.12) to be nonnegative:

$$c_1^2 - 4c_0c_2 \geqslant 0. \tag{3.13}$$

We say an C^1 Hermite data $\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0, \mathbf{D}_1$ *an admissible configuration* for (direct) C^1 Hermite interpolation using MPH quartic if the above discriminant is nonnegative. See Appendix B for underlying geometry.

To get a solution for C^1 Hermite interpolation problem, we suggest *a subdivision scheme*: If there exist no real solution for r, then add a point $\mathbf{P}_{1/2}$ and tangent vector $\mathbf{D}_{1/2}$ at $t = 1/2$ and solve two C^1 Hermite interpolation problems:

$$\alpha_1(t){:}\quad \alpha_1(0) = \mathbf{P}_0,\quad \alpha_1(1) = \mathbf{P}_{1/2},\quad \alpha_1'(0) = \mathbf{D}_0,\quad \alpha_1'(1) = \mathbf{D}_{1/2},$$
$$\alpha_2(t){:}\quad \alpha_2(0) = \mathbf{P}_{1/2},\quad \alpha_2(1) = \mathbf{P}_1,\quad \alpha_2'(0) = \mathbf{D}_{1/2},\quad \alpha_2'(1) = \mathbf{D}_1.$$

We take $\mathbf{P}_{1/2}$ and $\mathbf{D}_{1/2}$ as the midpoint of $\mathbf{P}_0, \mathbf{P}_1$ and $\mathbf{D}_0, \mathbf{D}_1$, respectively. Note that the subdivision must be done recursively and adaptively. If there exists a solution for r, hence a MPH quartic, then subdivision is useless.

3.1. Numerical results

Data are generated by random number generator fixing $\mathbf{P}_0 = (0,0,0)$. Only space-like vectors are chosen as tangent vectors $\mathbf{D}_0, \mathbf{D}_1$. First example data are as follows:

$$\mathbf{P}_0 = (0,0,0),\qquad \mathbf{D}_0 = (0.9934, 0.07224, 0.8761),$$
$$\mathbf{P}_1 = (0.6098, 0.4777, 0.6977),\qquad \mathbf{D}_1 = (0.0659, 0.2298, 0.0780).$$

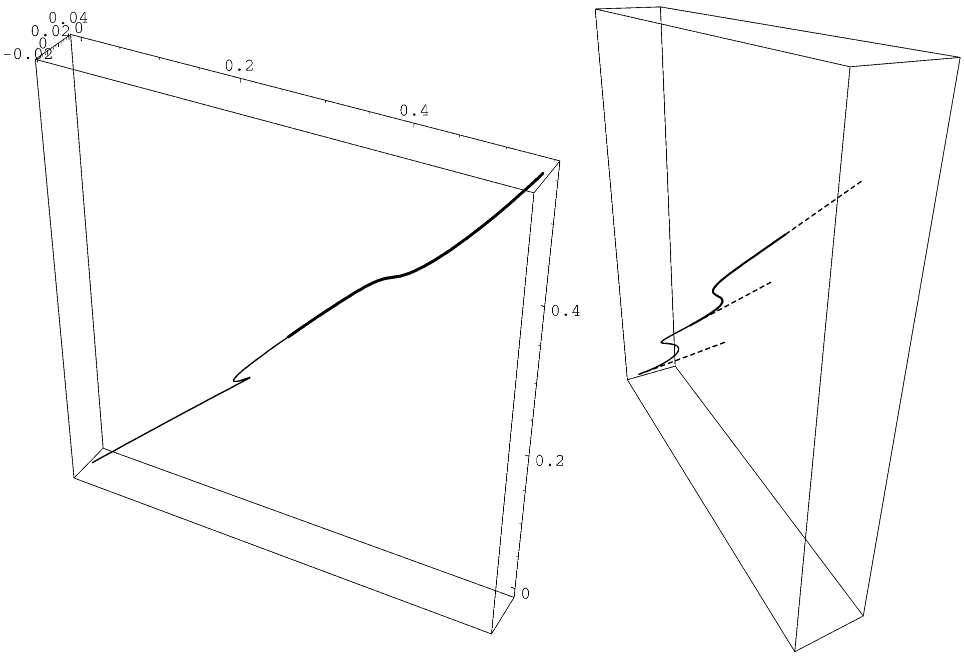

Fig. 2. A result of subdivision scheme. We divide once. Light curve is first segment and dark curve is second segment. Right figure is the same curves in a different view point. Dotted lines are tangent vectors.

Fig. 1 shows eight quartics obtained by Algorithm C^1 *Hermite interpolation using MPH quartic*. Dotted lines are tangent vectors and solid curves are interpolating quartic curves. Second data are as follows:

$$\mathbf{P}_0 = (0, 0, 0), \qquad \mathbf{D}_0 = (0.8299, 0.3331, 0.6852),$$
$$\mathbf{P}_1 = (0.5425, 0.0361, 0.5588), \qquad \mathbf{D}_1 = (0.3227, 0.3431, 0.4265).$$

In this case, $D = c_1^2 - 4c_0c_2$ is negative. Hence we subdivide once to get $\mathbf{P}_{1/2} = \frac{1}{2}(\mathbf{P}_0 + \mathbf{P}_1)$ and $\mathbf{D}_{1/2} = \frac{1}{2}(\mathbf{D}_0 + \mathbf{D}_1)$. Discriminants for these two small problems are positive and we get solutions. Fig. 2 shows the result. Light curve is the first segment interpolating $\mathbf{P}_0, \mathbf{D}_0, \mathbf{P}_{1/2}, \mathbf{D}_{1/2}$ and dark curve is the second segment interpolating $\mathbf{P}_{1/2}, \mathbf{D}_{1/2}, \mathbf{P}_1, \mathbf{D}_1$. Right figure is the same curve with a different view point. Dotted lines in right figure are tangent vectors.

4. $C^{1/2}$ interpolation

Note that G^1 interpolation using MPH cubic is completely solved (Choi et al., 1999). However, C^1 Hermite interpolation using MPH quartic is not always possible and there exists an admissible configuration of Hermite data presented in Appendix B. Now we will introduce a new concept, $C^{1/2}$ interpolation. By $C^{1/2}$ interpolation, we mean that given data $\mathbf{P}_0, \mathbf{P}_1, \mathbf{D}_0, \mathbf{D}_1$, to find a curve $\alpha(t)$ such that

$$\begin{aligned} &\alpha(0) = \mathbf{P}_0, \qquad \alpha'(0) = \mathbf{D}_0, \\ &\alpha(1) = \mathbf{P}_1, \qquad \alpha'(1) = \varepsilon\mathbf{D}_1 \end{aligned} \tag{4.1}$$

for some nonzero real number ε. That is an intermediate version of G^1 and C^1 interpolation.

From the Hermite data, we get the following constraints:

$$\mathbf{z}_1 = \mathbf{k}\big(\tfrac{1}{4} - \tfrac{1}{3}S_1 + \tfrac{1}{2}S_2 - S_3\big), \tag{4.2}$$

$$\mathbf{d}_0 = -\mathbf{k}S_3, \tag{4.3}$$

$$\varepsilon\mathbf{d}_1 = \mathbf{k}(1 - S_1 + S_2 - S_3), \tag{4.4}$$

$$\boldsymbol{\beta}_1 = \mathrm{e}^{i\theta}\mathbf{k}\big(\tfrac{1}{4} - \tfrac{1}{3}S_1^* + \tfrac{1}{2}S_2^* - S_3^*\big), \tag{4.5}$$

$$\tilde{\mathbf{d}}_0 = -\mathrm{e}^{i\theta}\mathbf{k}S_3^*, \tag{4.6}$$

$$\varepsilon\tilde{\mathbf{d}}_1 = \mathrm{e}^{i\theta}\mathbf{k}(1 - S_1^* + S_2^* - S_3^*), \tag{4.7}$$

where $\tilde{\mathbf{d}}_j = e_j \pm i\sqrt{\|\mathbf{d}_j\|^2 - e_j^2}$ for $j = 1, 2$ and S_j and S_j^* are the jth symmetric polynomials over $\{\omega_j\}$ and $\{\omega_j^*\}$, respectively.

Note that it suffice to modify case II in the (direct) C^1 Hermite interpolation presented in Section 3. From Eqs. (4.3), (4.4), (4.6), and (4.7), we get

$$\frac{\mathbf{d}_0}{\tilde{\mathbf{d}}_0} = \mathrm{e}^{-i\theta}\frac{\omega_1}{\overline{\omega}_1}, \qquad \frac{\mathbf{d}_1}{\tilde{\mathbf{d}}_1} = \mathrm{e}^{-i\theta}\frac{(1-\omega_1)}{(1-\overline{\omega}_1)}.$$

Dividing above two equations, we get

$$\frac{\mathbf{a}_1}{\mathbf{a}_0} = \frac{\bar{\lambda}}{\lambda},$$

where $\mathbf{a}_j = \mathbf{d}_j/\tilde{\mathbf{d}}_j$ for $j = 1, 2$ and $\lambda = \frac{1-\omega_1}{\omega_1}$. Note that λ is written as

$$\lambda = r\mathrm{e}^{i\eta}, \qquad \eta = \frac{\theta_1 - \theta_0}{2},$$

where $\theta_j = \arg(\mathbf{a}_j)$ for $j = 1, 2$. Thus,

$$\omega_1 = \frac{1}{1 + r\mathrm{e}^{i\eta}}.$$

Note that ω_1 runs on the perimeter of a circle which passes 0 and 1 and this circle is completely determined by $\mathbf{a}_0, \mathbf{a}_1$ since r is the only variable in ω_1.

Once we find ω_1, it is quite easy to find ω_2 and ω_3. From Eqs. (4.2)–(4.4), we get

$$\omega_1(\omega_2\omega_3)\mathbf{z}_1 = -\mathbf{d}_0\big(\tfrac{1}{4} - \tfrac{1}{3}(\omega_1 + \omega_2 + \omega_3) + \tfrac{1}{2}\big(\omega_1(\omega_2 + \omega_3) + \omega_2\omega_3\big) - \omega_1(\omega_2\omega_3)\big),$$

$$\varepsilon\omega_1(\omega_2\omega_3)\mathbf{d}_1 = -\mathbf{d}_0\big(1 - (\omega_1 + \omega_2 + \omega_3) + \omega_1(\omega_2 + \omega_3) + \omega_2\omega_3 - \omega_1(\omega_2\omega_3)\big).$$

Let T_1 and T_2 be the symmetric polynomials over ω_2 and ω_3. Then we get the following linear equation for T_1, T_2:

$$\begin{pmatrix} \omega_1\mathbf{z}_1 + \mathbf{d}_0(\frac{1}{2} - \omega_1) & \mathbf{d}_0(-\frac{1}{3} + \frac{1}{2}\omega_1) \\ \varepsilon\omega_1\mathbf{d}_1 + \mathbf{d}_0(1 - \omega_1) & -\mathbf{d}_0(1 - \omega_1) \end{pmatrix} \begin{pmatrix} T_2 \\ T_1 \end{pmatrix} = \begin{pmatrix} \mathbf{d}_0(-\frac{1}{4} + \frac{1}{3}\omega_1) \\ -\mathbf{d}_0(1 - \omega_1) \end{pmatrix}.$$

The determinant of the above matrix is given as follows:

$$\mathrm{Det}(r) = -\tfrac{1}{6}d_0(1 - \omega_1)(1 - 3\omega_1) - z_1\omega_1(1 - \omega_1) + \tfrac{1}{6}\varepsilon d_1\omega_1(2 - 3\omega_1). \tag{4.8}$$

Under an assumption $\mathrm{Det}(r_1) \neq 0$, we can find T_1, T_2 as follows:

$$T_2 = -\frac{1}{2}\frac{\mathbf{d}_0(1-3\omega_1+2\omega_1^2)}{\varepsilon\,\mathbf{d}_1(2\omega_1-3\omega_1^2)-\mathbf{d}_0(1-4\omega_1+3\omega_1^2)-6\mathbf{z}_1(\omega_1-\omega_1^2)},$$

$$T_1 = -\frac{1}{2}\frac{\varepsilon\,\mathbf{d}_1(3\omega_1-4\omega_1^2)-12\mathbf{z}_1(\omega_1-\omega_1^2)-\mathbf{d}_0(3-11\omega_1+8\omega_1^2)}{\varepsilon\,\mathbf{d}_1(2\omega_1-3\omega_1^2)-\mathbf{d}_0(1-4\omega_1+3\omega_1^2)-6\mathbf{z}_1(\omega_1-\omega_1^2)}.$$

$\mathbf{k}$ is given by $\mathbf{k} = -\frac{\mathbf{d}_0}{\omega_1 T_2}$. Plugging theses quantities into (4.5), we get

$$\begin{aligned}\boldsymbol{\beta}_1 &= \mathrm{e}^{i\theta}\mathbf{k}\left(\tfrac{1}{4}-\tfrac{1}{3}S_1^*+\tfrac{1}{2}S_2^*-S_3^*\right)\\ &= -\frac{\tilde{\mathbf{d}}_0}{\overline{\omega}_1 T_2}\left(\tfrac{1}{4}-\tfrac{1}{3}(\overline{\omega}_1+T_1)+\tfrac{1}{2}(\overline{\omega}_1 T_1+T_2)-\overline{\omega}_1 T_2\right)\\ &= \frac{N(\omega_1,\varepsilon)}{D(\omega_1)},\end{aligned}$$

where

$$\begin{aligned}N(\omega_1,\varepsilon) &= \varepsilon d_1\tilde{d}_0\omega_1(\overline{\omega_1}-\omega_1)+\tilde{d}_0\big(6z_1\omega_1(1-2\overline{\omega_1}-\omega_1+2\omega_1\overline{\omega_1})\\ &\quad + d_0\big(\overline{\omega_1}-\omega_1+\omega_1^2-\omega_1\overline{\omega_1}\big)\big) \qquad (4.9)\\ D(\omega_1) &= 6\overline{\omega_1}d_0\big(1-3\omega_1+2\omega_1^2\big) = 6\overline{\omega_1}d_0(1-\omega_1)(1-2\omega_1). \qquad (4.10)\end{aligned}$$

To satisfy the $C^{1/2}$ interpolation condition, we must find ε such that $r_1 = \mathrm{Re}(\beta_1)$. β_1 is given as follows:

$$\beta_1 = \frac{N(\omega_1,\varepsilon)}{D(\omega_1)} = A\varepsilon + B,$$

where

$$A = \frac{d_1\tilde{d}_0\omega_1(\overline{\omega_1}-\omega_1)}{D(\omega_1)}, \qquad (4.11)$$

$$B = \frac{N(\omega_1,\varepsilon)-\varepsilon d_1\tilde{d}_0\omega_1(\overline{\omega_1}-\omega_1)}{D(\omega_1)}. \qquad (4.12)$$

(Note that the numerator of B does not have any ε term.) Thus

$$r_1 = \mathrm{Re}(\beta_1) = \mathrm{Re}(A)\varepsilon + \mathrm{Re}(B). \qquad (4.13)$$

By choosing r such that $\mathrm{Re}(A) \neq 0$, $\mathrm{Re}(B) \neq r_1$ and $\mathrm{Det}(r) \neq 0$, ε is determined.

Remark 13. There are at most five singular solutions for r and henceforth for ω_1 (see Appendix C). Hence for all r except only at most five singular solutions, we can achieve $C^{1/2}$ interpolation.

5. Two step C^1 interpolation using MPH quartic

As we know, one step C^1 Hermite interpolation using MPH quartic is not possible in general. In this section, we present a two step C^1 Hermite interpolation scheme. Let $\mathbf{z}_1$, r_1, $\mathbf{d}_0$, $\mathbf{d}_1$, $\tilde{\mathbf{d}}_0$ and $\tilde{\mathbf{d}}_1$ be C^1 data with an assumption that the initial position is the origin. Choose an appropriate position $(\mathbf{z}_*, r_*)$ between the origin and $(\mathbf{z}_1, r_1)$ and an appropriate velocity $\mathbf{d}_*$. ($\tilde{\mathbf{d}}_*$ is determined automatically from these data.) As you will see in the following argument, the choices for $\mathbf{z}_*$, r_* and $\mathbf{d}_*$ are almost free. For example, we can set

$$r_* = \frac{r_1}{2}, \qquad \mathbf{z}_* = \frac{\mathbf{z}_1}{2}, \qquad \mathbf{d}_* = \frac{\mathbf{d}_0 + \mathbf{d}_1}{2}.$$

Now, consider two $C^{1/2}$ interpolations—a $C^{1/2}$ interpolation from the origin to the new mid point and a $C^{1/2}$ interpolation from the terminal point of the initial data to the new mid point. According to the previous argument in Section 4, we can find $C^{1/2}$ interpolants for each case. The key point is how to make ε in two interpolation problem coincide.

To complete this, first consider Eq. (4.13) again. Assume that ε is given. Then the only variable of Eq. (4.13) is r.

Using $\lambda = r\mathrm{e}^{\eta i}$, $\omega_1 = \frac{1}{r\mathrm{e}^{\eta i}+1}$, from (4.10) and (4.11), we get

$$A = \frac{m}{3}\frac{\mathbf{z}_1^*}{r\mathrm{e}^{\eta i} - 1} i = \frac{m}{3}\frac{\mathbf{z}_1^*(r\mathrm{e}^{-\eta i} - 1)}{(r\mathrm{e}^{\eta i} - 1)(r\mathrm{e}^{-\eta i} - 1)} i, \tag{5.1}$$

where $\mathbf{z}_1^* = \frac{\mathbf{d}_1\tilde{\mathbf{d}}_0\mathrm{e}^{-\eta i}}{\mathbf{d}_0}$ and $m = \sin(\eta)$. From (4.9), (4.10) and (4.12), we have

$$B = \frac{6\mathbf{z}_2^*(r\mathrm{e}^{-\eta i} - 1) + r(-\mathrm{e}^{-\eta i} + \mathrm{e}^{\eta i})}{6(r\mathrm{e}^{\eta i} - 1)}, \tag{5.2}$$

where $\mathbf{z}_2^* = \frac{\tilde{\mathbf{d}}_0 z_1}{\mathbf{d}_0}$ (for details, see Appendix C). Using (5.1) and (5.2), we get a quadratic equation equivalent to (4.13) as follows:

$$E_2 r^2 + E_1(\varepsilon) r + E_0(\varepsilon) = 0, \tag{5.3}$$

where

$$E_1 = M_1 + M_2\varepsilon, \qquad E_0 = M_3 + M_4\varepsilon$$

for some constants M_1, M_2, M_3, M_4 computed from given data and E_2 is also a constant without parameter ε. Assume that $E_2, M_2 \neq 0$. Then the discriminant $E_1^2 - 4E_2E_0$ of (5.3) is a quadratic equation in ε and the coefficient of ε^2 is positive. Therefore for ε sufficiently large, $E_1^2 - 4E_2E_0 \geqslant 0$, i.e., (5.3) has roots.

Remark 14. Since Eq. (5.3) has roots if its discriminant $E_1^2 - 4E_2E_0 \geqslant 0$ (when $E_2 \neq 0$), even though ε is not sufficiently large, there are possibilities that (5.3) has roots. For example, for ε satisfying $E_2E_0 < 0$, (5.3) has roots.

Next, consider a $C^{1/2}$ interpolation from the origin to the new mid point. For given data $\mathbf{z}_*, r_*, \mathbf{d}_0, \tilde{\mathbf{d}}_0, \mathbf{d}_*, \tilde{\mathbf{d}}_*$, we solve the following equation:

$$r_* = \mathrm{Re}\big(A_1(\omega_1^*)\big)\varepsilon + \mathrm{Re}\big(B_1(\omega_1^*)\big). \tag{5.4}$$

For second interpolant, we solve the following equation from the data $\mathbf{z}_* - \mathbf{z}_1, r_* - r_1, -\mathbf{d}_1, -\tilde{\mathbf{d}}_1, -\mathbf{d}_*, -\tilde{\mathbf{d}}_*$:

$$r_* - r_1 = \mathrm{Re}\big(A_2(\omega_1^{**})\big)\varepsilon + \mathrm{Re}\big(B_2(\omega_1^{**})\big). \tag{5.5}$$

Note that for each of (5.4) and (5.5), if we assume that all data are given except d_*, then only finite d_*s satisfy $E_2 = 0$ or $M_2 = 0$. By the previous argument and Remark 14, for d_* except these solutions, we can get two $C^{1/2}$ interpolants with same scaling factor ε. That is, we get an C^1 interpolant from the origin to the new mid point $(\mathbf{z}_*, r_*)$ with the velocity data $\mathbf{d}_0, \tilde{\mathbf{d}}_0, \varepsilon\mathbf{d}_*, \varepsilon\tilde{\mathbf{d}}_*$ and another C^1 interpolant from the new mid point $(\mathbf{z}_*, r_*)$ to the initial end point $(\mathbf{z}_1, r_1)$ with the velocity data $\varepsilon\mathbf{d}_*, \varepsilon\tilde{\mathbf{d}}_*, \mathbf{d}_1, \tilde{\mathbf{d}}_1$, respectively.

Remark 15. There may exist at most 10 singular choices for $r - 5$ from each divided $C^{1/2}$ interpolation. However, these 10 singular cases can be predetermined by the input data.

Algorithm (*Two step C^1 Hermite interpolation using MPH quartic*)
Input: $P_0, P_1, D_0, D_1 \in \mathbb{R}^{2,1}$
Output: MPH quartic
($*$ D_0, D_1 must be space-like vectors $*$)

1. Compute $\mathbf{z}_0, \mathbf{z}_1, \mathbf{d}_0, \mathbf{d}_1, \tilde{\mathbf{d}}_0, \tilde{\mathbf{d}}_1$.
2. ($*$ We have four cases according to the sign of imaginary parts of $\tilde{\mathbf{d}}_0, \tilde{\mathbf{d}}_1$. $*$)
3. $\mathbf{z}_1 \leftarrow \mathbf{z}_1 - \mathbf{z}_0$
4. **for** each 4 cases
5.   **do** $\mathbf{a}_0 \leftarrow \frac{\mathbf{d}_0}{\tilde{\mathbf{d}}_0}, \mathbf{a}_1 \leftarrow \frac{\mathbf{d}_1}{\tilde{\mathbf{d}}_1}$
6.   **if** $\mathbf{a}_0 = \mathbf{a}_1$
7.   **then if** $r_1 = \mathrm{Re}(\mathbf{z}_1/\mathbf{a}_0)$
8.   **then** solve the plane cubic Hermite interpolation problem
9.   **else qreturn** "There are no solution."
10.   **else if** admissible configuration
11.   **then** solve case II in (one step) C^1 interpolation
12.   **else** make a mid point and solve two $C^{1/2}$ interpolation.

Note that for the (adaptive) subdivision scheme, 12 in the above algorithm 15 should be changed into "*make a mid point and solve two C^1 interpolation problem*".

5.1. Numerical results

We apply our two step C^1 Hermite interpolation scheme to a data:

$$P_0 = (9.43474, 19.1296, 0.259429), \qquad P_1 = (19.3208, 21.4930, 0.375256),$$
$$D_0 = (10.0124, 19.2597, 0.570228), \qquad D_1 = (12.6065, 14.4561, 0.024123).$$

Fig. 3 shows the resulting interpolant. Dotted lines are scaled tangent vectors, light curve is the first segment and bold curve is the second segment. Also the left figure in Fig. 4 shows the radius (t versus r) and spine (x versus y) plot. The right-hand side figure in Fig. 4 shows the envelope when we interpret the interpolant as a curve representation of one parameter family of circles.

See Figs. 5 and 6 for the second data set:

$$P_0 = (12.8242, 5.87048, 0.164317), \qquad P_1 = (20.2475, 6.26066, 0.735760),$$
$$D_0 = (4.19536, 17.9409, 0.046602), \qquad D_1 = (12.1594, 10.6271, 0.021634).$$

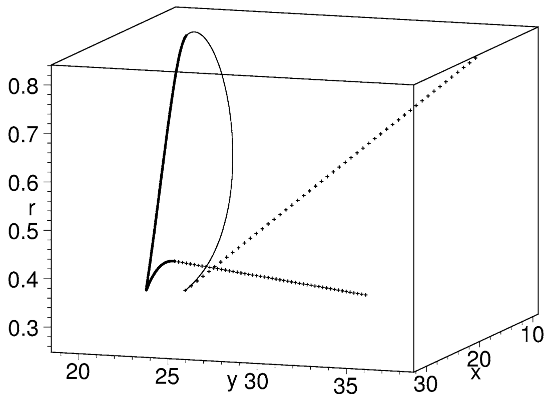

Fig. 3. A MPH quartic C^1 Hermite interpolant. Dotted lines are given tangent vectors, solid light curve is the first segment and solid bold curve is the second segment.

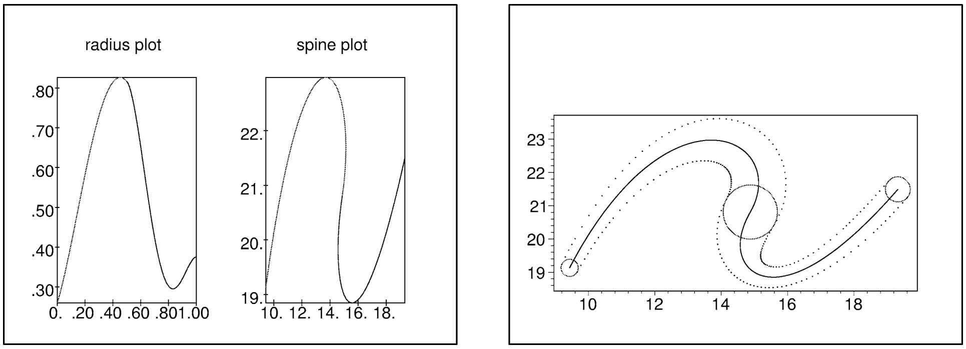

Fig. 4. Left: Radius and spine plot. Right: Resulting envelope.

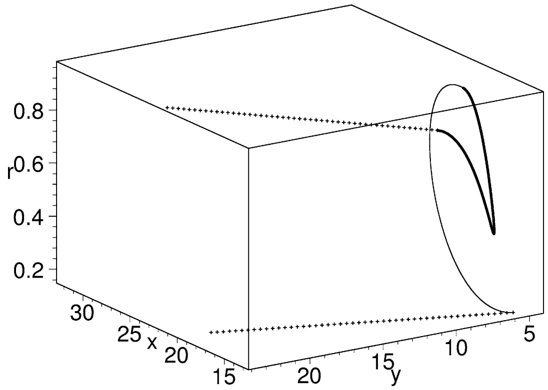

Fig. 5. A MPH quartic C^1 Hermite interpolant. Dotted lines are given tangent vectors, solid light curve is the first segment and solid bold curve is are the second segment.

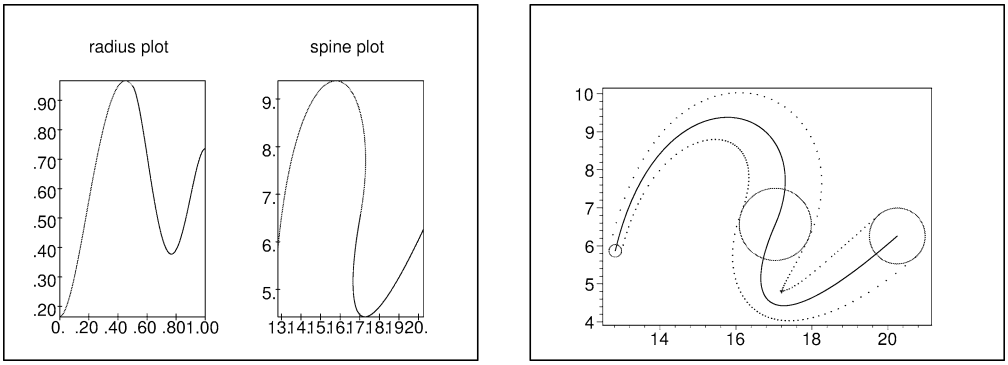

Fig. 6. Left: Radius and spine plot. Right: Resulting envelope plot.

Remark 16. From the fact that the success of two step C^1 interpolation is guaranteed with probability 1, we can infer that the (adapted) subdivision scheme is a modified version of the two step method.

6. Concluding remark and further studies

We characterized MPH curves in $\mathbb{R}^{2,1}$: All MPH curves are determined by the roots of hodographs of their complexified spine curves and some conjugates of roots. In other words, for a MPH curve $\alpha(t) = (x(t), y(t), r(t))$, we proved that a curve $\alpha(t)$ is MPH if and only if the complexified curve $r'(t) + \sigma(t)i$ where $\sigma(t)$ satisfies $x'(t)^2 + y'(t)^2 = r'(t)^2 + \sigma(t)^2$ is obtained by rotating the curve whose zeros are semi-equal to those of $x'(t) + y'(t)i$ and the envelope generated by the MPH $\alpha(t)$ is a rational PH curve. C^1 Hermite interpolation problem with MPH quartic curve was also dealt. To overcome the difficulty in guaranteeing the existence of solutions, we introduced $C^{1/2}$ Hermite interpolation and showed that two step C^1 Hermite interpolation is possible.

As further studies, we introduce two research directions regarding the choices. First is the choices for intermediate data. In two step interpolation scheme, as we pointed out at the beginning of Section 5, the choices for intermediate point and its speed data are almost free. As it is well known that polynomial curves of high degree contain wiggles, bad choice for $\mathbf{d}_*$ is apt to lead wiggles in the resulting interpolant. For this issue, we need more insight on the shape of polynomial curves and variety of numerical data with varying intermediate data. Second is the choice for r_*. To get the best choice for r_*, we need some criteria on the quality of the resulting interpolant. Minimization of curvature variation of interpolant may be an answer for this best choice problem. Currently, we are tackling this problem in this direction.

Appendix A. Coefficients of a quadratic equation

After plugging all data into Eq. (3.11), we get the following quadratic equation in r:

$$c_0 + c_1 r + c_2 r^2 = 0,$$

where the coefficients c_j are given as follows:

$$
\begin{aligned}
2c_0 = & -(\tilde{d}_{0x} d_{0x} d_{1x}) - \tilde{d}_{0y} d_{0y} d_{1x} + \tilde{d}_{0y} d_{0x} d_{1y} - \tilde{d}_{0x} d_{0y} d_{1y} \\
& - 6d_{0x}^2 r_1 - 6d_{0y}^2 r_1 + 6\tilde{d}_{0x} d_{0x} z_{1x} + 6\tilde{d}_{0y} d_{0y} z_{1x} - 6\tilde{d}_{0y} d_{0x} z_{1y} + 6\tilde{d}_{0x} d_{0y} z_{1y} \\
& + \tilde{d}_{0x} d_{0x} d_{1x} \cos(2\eta) + \tilde{d}_{0y} d_{0y} d_{1x} \cos(2\eta) - \tilde{d}_{0y} d_{0x} d_{1y} \cos(2\eta) + \tilde{d}_{0x} d_{0y} d_{1y} \cos(2\eta) \\
& + \tilde{d}_{0y} d_{0x} d_{1x} \sin(2\eta) - \tilde{d}_{0x} d_{0y} d_{1x} \sin(2\eta) + \tilde{d}_{0x} d_{0x} d_{1y} \sin(2\eta) + \tilde{d}_{0y} d_{0y} d_{1y} \sin(2\eta),
\end{aligned}
$$

$$
\begin{aligned}
2c_1 = & \tilde{d}_{0x} d_{0x} d_{1x} \cos(\eta) + \tilde{d}_{0y} d_{0y} d_{1x} \cos(\eta) - \tilde{d}_{0y} d_{0x} d_{1y} \cos(\eta) + \tilde{d}_{0x} d_{0y} d_{1y} \cos(\eta) \\
& + 12 d_{0x}^2 r_1 \cos(\eta) + 12 d_{0y}^2 r_1 \cos(\eta) - 12\tilde{d}_{0x} d_{0x} z_{1x} \cos(\eta) - 12\tilde{d}_{0y} d_{0y} z_{1x} \cos(\eta) \\
& + 12\tilde{d}_{0y} d_{0x} z_{1y} \cos(\eta) - 12\tilde{d}_{0x} d_{0y} z_{1y} \cos(\eta) - \tilde{d}_{0x} d_{0x} d_{1x} \cos(3\eta) - \tilde{d}_{0y} d_{0y} d_{1x} \cos(3\eta) \\
& + \tilde{d}_{0y} d_{0x} d_{1y} \cos(3\eta) - \tilde{d}_{0x} d_{0y} d_{1y} \cos(3\eta) + 2\tilde{d}_{0y} d_{0x}^2 \sin(\eta) + 2\tilde{d}_{0y} d_{0y}^2 \sin(\eta) \\
& + \tilde{d}_{0y} d_{0x} d_{1x} \sin(\eta) - \tilde{d}_{0x} d_{0y} d_{1x} \sin(\eta) + \tilde{d}_{0x} d_{0x} d_{1y} \sin(\eta) + \tilde{d}_{0y} d_{0y} d_{1y} \sin(\eta) \\
& - 12\tilde{d}_{0y} d_{0x} z_{1x} \sin(\eta) + 12\tilde{d}_{0x} d_{0y} z_{1x} \sin(\eta) - 12\tilde{d}_{0x} d_{0x} z_{1y} \sin(\eta) - 12\tilde{d}_{0y} d_{0y} z_{1y} \sin(\eta) \\
& - \tilde{d}_{0y} d_{0x} d_{1x} \sin(3\eta) + \tilde{d}_{0x} d_{0y} d_{1x} \sin(3\eta) - \tilde{d}_{0x} d_{0x} d_{1y} \sin(3\eta) - \tilde{d}_{0y} d_{0y} d_{1y} \sin(3\eta),
\end{aligned}
$$

$$
\begin{aligned}
2c_2 = & \tilde{d}_{0x} d_{0x}^2 + \tilde{d}_{0x} d_{0y}^2 - 6d_{0x}^2 r_1 - 6d_{0y}^2 r_1 - \tilde{d}_{0x} d_{0x}^2 \cos(2\eta) - \tilde{d}_{0x} d_{0y}^2 \cos(2\eta) \\
& + 6\tilde{d}_{0x} d_{0x} z_{1x} \cos(2\eta) + 6\tilde{d}_{0y} d_{0y} z_{1x} \cos(2\eta) - 6\tilde{d}_{0y} d_{0x} z_{1y} \cos(2\eta) \\
& + 6\tilde{d}_{0x} d_{0y} z_{1y} \cos(2\eta) - \tilde{d}_{0y} d_{0x}^2 \sin(2\eta) - \tilde{d}_{0y} d_{0y}^2 \sin(2\eta) + 6\tilde{d}_{0y} d_{0x} z_{1x} \sin(2\eta) \\
& - 6\tilde{d}_{0x} d_{0y} z_{1x} \sin(2\eta) + 6\tilde{d}_{0x} d_{0x} z_{1y} \sin(2\eta) + 6\tilde{d}_{0y} d_{0y} z_{1y} \sin(2\eta),
\end{aligned}
$$

where $\mathbf{d}_0 = d_{0x} + i d_{0y}$, $\mathbf{d}_1 = d_{1x} + i d_{1y}$, $\tilde{\mathbf{d}}_0 = \tilde{d}_{0x} + i\tilde{d}_{0y}$, $\tilde{\mathbf{d}}_1 = \tilde{d}_{1x} + i\tilde{d}_{1y}$ and $\mathbf{z}_1 = z_{1x} + i z_{1y}$. To compute the above coefficients, we use Mathematica. After some manipulations, we get compact form:

$$
\begin{aligned}
2c_0 &= -6\|\mathbf{d}_0\|^2 r_1 + 6\,\mathrm{Re}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{z}_1) + \mathrm{Re}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{d}_1)\big(\cos(2\eta) - 1\big) + \mathrm{Im}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{d}_1)\sin(2\eta), \\
2c_1 &= \mathrm{Re}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{d}_1)\cos(\eta) + 12\|\mathbf{d}_0\|^2 r_1 \cos(\eta) - 12\,\mathrm{Re}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{z}_1)\cos(\eta) - \mathrm{Re}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{d}_1)\cos(3\eta) \\
&\quad + 2\,\mathrm{Im}(\tilde{\mathbf{d}}_0)\|\mathbf{d}_0\|^2\sin(\eta) + \mathrm{Im}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{d}_1)\sin(\eta) - 12\,\mathrm{Im}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{z}_1)\sin(\eta) - \mathrm{Im}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{d}_1)\sin(3\eta), \\
2c_2 &= \mathrm{Re}(\mathbf{d}_0)\|\mathbf{d}_0\|^2 - 6\|\mathbf{d}_0\|^2 r_1 - \mathrm{Re}(\tilde{\mathbf{d}}_0)\|\mathbf{d}_0\|^2\cos(2\eta) + 6\,\mathrm{Re}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{z}_1)\cos(2\eta) \\
&\quad - \mathrm{Im}(\tilde{\mathbf{d}}_0)\|\mathbf{d}_0\|^2\sin(2\eta) + 6\,\mathrm{Im}(\bar{\mathbf{d}}_0\tilde{\mathbf{d}}_0\mathbf{z}_1)\sin(2\eta),
\end{aligned}
$$

equivalently,

$$
\begin{aligned}
2c_0 &= -6r_1 + \mathrm{Re}\big(\mathbf{a}_0^{-1}\big(6\mathbf{z}_1 + \mathbf{d}_1 \mathrm{e}^{-2i\eta} - \mathbf{d}_1\big)\big), \\
2c_1 &= 12r_1\cos(\eta) + \mathrm{Re}\big(\mathbf{a}_0^{-1}\big(\mathbf{d}_1\mathrm{e}^{-i\eta} - 12\mathbf{z}_1\mathrm{e}^{-i\eta} - \mathbf{d}_1\mathrm{e}^{-3i\eta}\big)\big) + 2\,\mathrm{Im}(\tilde{\mathbf{d}}_0)\sin(\eta), \\
2c_2 &= -6r_1 + \mathrm{Re}\big(\mathbf{a}_0^{-1}\big(\mathbf{d}_0 - \mathbf{d}_0\mathrm{e}^{-2i\eta} + 6\mathbf{z}_1\mathrm{e}^{-2i\eta}\big)\big).
\end{aligned}
$$

Appendix B. Existence of solution

To guarantee the existence of solution, we need the discriminant of the quadratic equation (3.12) to be nonnegative:

$$c_1^2 - 4c_0c_2 \geqslant 0. \tag{B.1}$$

Introduce new symbols α, β, γ such that

$$c_0 = -3(r_1 - \alpha), \qquad c_2 = -3(r_1 - \beta), \qquad c_1 = 6(r_1 \cos\eta + \gamma).$$

Then

$$D/4 = 9\left\{-\sin^2\eta r_1^2 + 2\left(\gamma\cos\eta - \frac{\alpha+\beta}{2}\right)r_1 + \gamma^2 - \alpha\beta\right\}.$$

Let r_1^+ and r_1^- be roots of $D/4 = 0$. Then the possible range of r_1 is a closed interval $[r_1^{-1}, r_1^+]$ if other data are fixed. The whole set of data $\mathbf{P}_1, \mathbf{D}_0, \mathbf{D}_1$ guaranteeing the existence of solution forms a manifold of dimension 9 in $\mathbb{R}^3 \times \mathcal{S} \times \mathcal{S}$, where $\mathcal{S}$ is the set of space-like vectors in $R^{2,1}$.

The coefficients c_0, c_1, c_2 are rewritten as follows:

$$2c_0 = -6r_1 + \left(6\,\mathrm{Re}\left(\frac{\mathbf{z}_1}{\mathbf{a}_0}\right) - \mathrm{Re}\left(\frac{\mathbf{d}_1}{\mathbf{a}_0} - \frac{\mathbf{d}_1}{\mathbf{a}_1}\right)\right),$$
$$2c_2 = -6r_1 + \left(6\,\mathrm{Re}\left(\frac{\mathbf{z}_1}{\mathbf{a}_1}\right) - \mathrm{Re}\left(\frac{\mathbf{d}_0}{\mathbf{a}_0} - \frac{\mathbf{d}_0}{\mathbf{a}_1}\right)\right),$$
$$2c_1 = 12\cos\eta r_1 - \left(12\,\mathrm{Re}\left(\frac{\mathbf{z}_1}{\mathbf{a}_0}\mathrm{e}^{-i\eta}\right) - \mathrm{Re}\left(\left(\frac{\mathbf{d}_1}{\mathbf{a}_0} - \frac{\mathbf{d}_1}{\mathbf{a}_0}\right)\mathrm{e}^{-i\eta}\right) + \mathrm{Im}\left(\frac{\mathbf{d}_0}{\mathbf{a}_0}\right)\sin\eta\right).$$

Let

$$\lambda_0 = -6r_1 + 6\frac{\mathbf{z}_1}{\mathbf{a}_0} - \left(\frac{\mathbf{d}_1}{\mathbf{a}_0} - \frac{\mathbf{d}_1}{\mathbf{a}_1}\right),$$
$$\lambda_2 = -6r_1 + 6\frac{\mathbf{z}_1}{\mathbf{a}_1} - \left(\frac{\mathbf{d}_0}{\mathbf{a}_0} - \frac{\mathbf{d}_0}{\mathbf{a}_1}\right).$$

Then $2c_0 = \mathrm{Re}(\lambda_0)$ and $2c_2 = \mathrm{Re}(\lambda_2)$. Consider a complex number $\frac{1}{2}(\lambda_0\mathrm{e}^{-i\eta} + \lambda_2\mathrm{e}^{i\eta})$.

$$-\mathrm{Re}\left(\frac{1}{2}\left(\lambda_0\mathrm{e}^{-i\eta} + \lambda_2\mathrm{e}^{i\eta}\right)\right) = -6r_1\cos\eta - 6\,\mathrm{Re}\left(\frac{\mathbf{z}_1}{\mathbf{a}_0}\mathrm{e}^{-i\eta}\right) + \frac{1}{2}\mathrm{Re}\left(\left(\frac{\mathbf{d}_1}{\mathbf{a}_0} - \frac{\mathbf{d}_1}{\mathbf{a}_1}\right)\mathrm{e}^{-i\eta}\right)$$
$$- \frac{1}{2}\mathrm{Re}\left(\left(\frac{\mathbf{d}_0}{\mathbf{a}_0} - \frac{\mathbf{d}_0}{\mathbf{a}_1}\right)\mathrm{e}^{i\eta}\right).$$

And,

$$-\frac{1}{2}\mathrm{Re}\left(\left(\frac{\mathbf{d}_0}{\mathbf{a}_0} - \frac{\mathbf{d}_0}{\mathbf{a}_1}\right)\mathrm{e}^{i\eta}\right) = -\frac{1}{2}\mathrm{Re}\left(\mathbf{d}_0\left(\frac{\mathbf{a}_1 - \mathbf{a}_0}{\mathbf{a}_0\mathbf{a}_1}\right)\mathrm{e}^{i\eta}\right) = -\frac{1}{2}\mathrm{Re}\left(\frac{\mathbf{d}_0}{\mathbf{a}_0}\frac{\mathrm{e}^{2i\eta} - 1}{\mathrm{e}^{i\eta}}\right)$$
$$= -\frac{1}{2}\mathrm{Re}\left(\frac{\mathbf{d}_0}{\mathbf{a}_0}2\sin\eta i\right) = \mathrm{Im}\left(\frac{\mathbf{d}_0}{\mathbf{a}_0}\right)\sin\eta.$$

Therefore, we get the following relations:

$$2c_0 = \mathrm{Re}(\lambda_0),$$
$$2c_1 = \mathrm{Re}\left(\lambda_0\mathrm{e}^{-i\eta} + \lambda_2\mathrm{e}^{i\eta}\right),$$
$$2c_2 = \mathrm{Re}(\lambda_2).$$

In other words,

$$2c_0 = \lambda_0^r,$$
$$2c_1 = \lambda_0^r \cos\eta + \lambda_0^i \sin\eta + \lambda_2^r \cos\eta - \lambda_2^i \sin\eta,$$
$$2c_2 = \lambda_2^r.$$

Assume that all data except r_1 are fixed. Then

$$c_2 = c_0 + N, \tag{B.2}$$

where $N = \frac{1}{2}\operatorname{Re}(\lambda_0 - \lambda_2) = \frac{1}{2}\operatorname{Re}((6\mathbf{z}_1 - \mathbf{d}_0 + \mathbf{d}_1)(\frac{1}{\mathbf{a}_1} - \frac{1}{\mathbf{a}_0}))$ is independent of r_1. Let $M = \frac{1}{2}(\lambda_0^i - \lambda_2^i)$. Note that M is independent of r_1. Then $c_1 = (c_0 + c_2)\cos\eta + M\sin\eta$. Consider

$$\begin{aligned} f(c_0, c_2) &= c_1^2 - 4c_0c_2 \\ &= \cos^2\eta\big(c_0^2 + c_2^2\big) + 2\big(\cos^2\eta - 2\big)c_0c_2 + 2\cos\eta(c_0 + c_2)M\sin\eta + M^2\sin^2\eta \\ &\geqslant 0. \end{aligned} \tag{B.3}$$

Thus we know that there exists a solution if and only if (c_0, c_2) lies on the intersection of the line (B.2) and the region (B.3). To view the geometry more easily, we rotate the line and the conic by $-45°$. That is,

$$\begin{pmatrix} c_0 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Thus the line and the conic are transformed into

$$\begin{aligned} u_2 &= \frac{1}{\sqrt{2}}N, \\ f(u_1, u_2) &= \cos^2\eta\big(u_1^2 + u_2^2\big) + \big(\cos^2\eta - 2\big)\big(u_1^2 - u_2^2\big) + 2\sqrt{2}\cos\eta\sin\eta M u_1 + \sin^2\eta M^2 \\ &= -2\sin^2\eta u_1^2 + 2u_2^2 + 2\sqrt{2}\cos\eta\sin\eta M u_1 + \sin^2\eta M^2 \\ &= -2\sin^2\eta(u_1 - K)^2 + 2u_2^2 + M^2 \\ &\geqslant 0, \end{aligned}$$

where $K = \frac{\cos\eta}{\sqrt{2}\sin\eta}M$. Fig. B.1 shows the hyperbola and the line. The line and the hyperbola meet at $u_1 = \alpha, \beta$. The line segment contained in the dashed region is the feasible region. Note that $(0, 0)$ is always contained in the region $f(u_1, u_2) \geqslant 0$. Thus, $\alpha < 0 < \beta$. If $u_1^* = \frac{1}{2\sqrt{2}}(-12r_1 + 6(\frac{\mathbf{z}_1}{\mathbf{a}_0} + \frac{\mathbf{z}_1}{\mathbf{a}_1}) - (\frac{\mathbf{d}_1}{\mathbf{a}_0} - \frac{\mathbf{d}_1}{\mathbf{a}_1}) - (\frac{\mathbf{d}_0}{\mathbf{a}_0} - \frac{\mathbf{d}_0}{\mathbf{a}_1}))$ lies between α and β, then the solution exists.

Appendix C. All possible singular cases

C.1. $\operatorname{Re}(A) = 0$

Let us consider the condition $\operatorname{Re}(A) \neq 0$ which is a necessary condition for (4.13) have a solution ε.

$$A = \frac{\mathbf{d}_1\tilde{\mathbf{d}}_0\omega_1(\overline{\omega_1} - \omega_1)}{D(\omega_1)} = \frac{\mathbf{d}_1\tilde{\mathbf{d}}_0\omega_1(\overline{\omega_1} - \omega_1)}{6\mathbf{d}_0\overline{\omega_1}(1 - \omega_1)(1 - 2\omega_1)}.$$

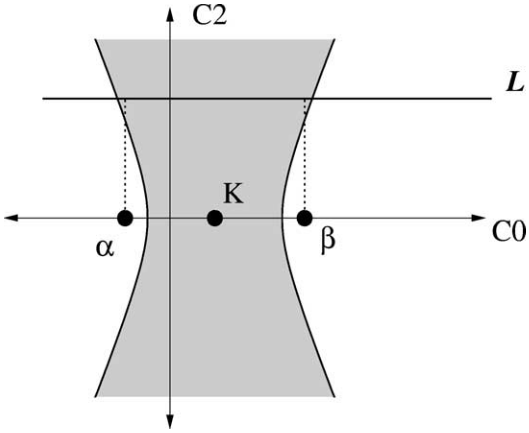

Fig. B.1. Hyperbola and line.

Using $\lambda = r\mathrm{e}^{\eta i}, \omega_1 = \frac{1}{r\mathrm{e}^{\eta i}+1}$, A is rewritten as follows:

$$A = \left(\frac{\mathbf{d}_1\tilde{\mathbf{d}}_0}{6\mathbf{d}_0}\right)\left(\frac{\omega_1}{1-\omega_1}\right)\left(\frac{\overline{\omega_1}-\omega_1}{\overline{\omega_1}(1-2\omega_1)}\right) = \frac{1}{6r}\left(\frac{\mathbf{d}_1\tilde{\mathbf{d}}_0\mathrm{e}^{-\eta i}}{\mathbf{d}_0}\right)\left(\frac{r\mathrm{e}^{\eta i}-r\mathrm{e}^{-\eta_i}}{r\mathrm{e}^{\eta i}-1}\right).$$

Let $\mathbf{z}_1^* = \frac{\mathbf{d}_1\tilde{\mathbf{d}}_0\mathrm{e}^{-\eta i}}{\mathbf{d}_0}$ and $m = \sin(\eta)$. Then

$$A = \frac{m}{3}\frac{\mathbf{z}_1^*}{r\mathrm{e}^{\eta i}-1}i = \frac{m}{3}\frac{\mathbf{z}_1^*(r\mathrm{e}^{-\eta i}-1)}{(r\mathrm{e}^{\eta i}-1)(r\mathrm{e}^{-\eta i}-1)}i.$$

Thus $\mathrm{Re}(A) = 0$ is equivalent to $\mathrm{Im}(\mathbf{z}_1^*(r\mathrm{e}^{-\eta i}-1)) = 0$. Let $b_1 = \mathrm{Im}(\mathbf{z}_1^*\mathrm{e}^{-\eta i})$ and $b_2 = \mathrm{Im}(\mathbf{z}_1^*)$. Then $\mathrm{Re}(A) = 0$ if and only if $r = b_2/b_1$. Thus we can choose r freely except $r = b_2/b_1$.

C.2. $\mathrm{Re}(B) = r_1$

First consider $\mathrm{Re}(B) = r_1$. Then we get $\varepsilon = 0$ and it implies that $\sqrt{x'(t)^2 + y'(t)^2}|_{t=1} = 0$, $\sqrt{z'(t)^2 + \sigma(t)^2}|_{t=1} = 0$. It contradicts to the regularity of MPH curve. Thus we must identify this case

$$\begin{aligned}
r_1 + \mathrm{Im}(B)i = B &= \frac{N(\omega_1,\varepsilon) - \varepsilon\mathbf{d}_1\tilde{\mathbf{d}}_0\omega_1(\overline{\omega_1}-\omega_1)}{D(\omega_1)} \\
&= \frac{6\tilde{\mathbf{d}}_0 z_1\omega_1(1-2\overline{\omega_1}) + \mathbf{d}_0(\overline{\omega_1}-\omega_1)}{6\mathbf{d}_0\overline{\omega_1}(1-2\omega_1)} \\
&= \mathbf{z}_2^*\frac{(1/\overline{\omega_1}-2)}{(1/\omega_1-2)} + \frac{(-1/\overline{\omega_1}+1/\omega_1)}{6(1/\omega_1-2)} \\
&= \mathbf{z}_2^*\frac{r\mathrm{e}^{-\eta i}-1}{r\mathrm{e}^{\eta i}-1} + \frac{-r\mathrm{e}^{-\eta i}+r\mathrm{e}^{\eta i}}{6(r\mathrm{e}^{\eta i}-1)} \\
&= \frac{6\mathbf{z}_2^*(r\mathrm{e}^{-\eta i}-1) + r(-\mathrm{e}^{-\eta i}+\mathrm{e}^{\eta i})}{6(r\mathrm{e}^{\eta i}-1)},
\end{aligned}$$

where $\mathbf{z}_2^* = \frac{\tilde{\mathbf{d}}_0 z_1}{\mathbf{d}_0}$. Thus taking the real part of the both part,

$$6r_1\big(r\mathrm{e}^{\eta i}-1\big)\big(r\mathrm{e}^{-\eta i}-1\big)=\mathrm{Re}\Big(6\mathbf{z}_2^*\big(r\mathrm{e}^{-\eta i}-1\big)^2+r\big(\mathrm{e}^{-\eta i}-\mathrm{e}^{\eta i}\big)\big(r\mathrm{e}^{-\eta i}-1\big)\Big),$$

equivalently,

$$\begin{aligned}&\Big(\mathrm{Re}\big(6\mathbf{z}_2^*\mathrm{e}^{-2\eta i}+\mathrm{e}^{-2\eta i}-1\big)-6r_1\Big)r^2\\&\quad-\Big(\mathrm{Re}\big(12\mathbf{z}_2^*\mathrm{e}^{-\eta i}\big)+\mathrm{Re}\big(\mathrm{e}^{-\eta i}-\mathrm{e}^{\eta i}\big)-6r_1\big(\mathrm{e}^{\eta i}+\mathrm{e}^{-\eta i}\big)\Big)r+\big(6\,\mathrm{Re}(z_2^*)-6r_1\big)=0.\end{aligned}$$

That is,

$$(a_1-6r_1)r^2-(a_2-6a_3r_1)r+6(a_4-r_1)=0.$$

Thus we have at most two singular solutions. Moreover, the coefficient of the above quadratic equation is predetermined by input data.

C.3. Det = 0

Second, what if Det = 0? From (4.8), a real number ε is given as follows:

$$\varepsilon=\frac{\mathbf{d}_0}{\mathbf{d}_1}\frac{(1/\omega_1-1)(1/\omega_1-3)-6\mathbf{z}_1/\mathbf{d}_0(1/\omega_1-1)}{2/\omega_1-3}=\mathbf{z}_4^* r\mathrm{e}^{\eta i}\frac{\mathrm{e}^{\eta i}-\mathbf{z}_3^*}{2\mathrm{e}^{\eta i}-1},$$

where $\mathbf{z}_3^*=2-6\mathbf{z}_1/\mathbf{d}_0$ and $\mathbf{z}_4^*=\mathbf{d}_0/\mathbf{d}_1$. Since ε is a real number, $\mathrm{Im}(\varepsilon)=0$:

$$\begin{aligned}0&=\mathrm{Im}\Big(r\big(\mathbf{z}_4^*\mathrm{e}^{\eta i}\big)\big(r\mathrm{e}^{\eta i}-z_3^*\big)\big(2r\mathrm{e}^{-\eta i}-1\big)\Big)\\&=r\Big(\mathrm{Im}\big(2\mathbf{z}_4^*\mathrm{e}^{\eta i}\big)r^2-\mathrm{Im}\big(\mathbf{z}_4^*\big(2z_3^*+\mathrm{e}^{2\eta i}\big)\big)r+\mathrm{Im}\big(\mathbf{z}_4^* z_3^*\mathrm{e}^{\eta i}\big)\Big)\\&=r\big(d_1r^2-d_2r+d_3\big).\end{aligned}$$

If $r=0$, then ω_1 is a real number and this also contradicts to initial assumption. Thus the singular solutions come from the quadratic equation $d_1r^2-d_2r+d_3=0$.

References

Ahn, M.-H., Kim, G.-I., t.a. Characterization of Pythagorean curves and Pythagoreanization using rational transform, J. Symbolic Comput., submitted.

Choi, H.I., Lee, D.S., 2000. Rational parametrization of canal surface by 4-dimensional Minkowski Pythagorean hodograph curves. In: Geometric Modeling and Processing 2000, Hong Kong, China, pp. 301–309.

Choi, H.I., Lee, D.S., Moon, H.P., 2002. Clifford algebra, spin representation, and rational parametrization of curves and surfaces. Adv. Comput. Math. 17, 5–48.

Choi, H.I., Han, C.Y., Moon, H.P., Roh, K.H., Wee, N.S., 1999. Medial axis transform and offset curves by Minkowski Pythagorean hodograph curves. Computer-Aided Design 31, 59–72.

Farouki, R.T., 1992. Pythagorean-hodograph curves in practical use. In: Barnhill, R.E. (Ed.), Geometry Processing for Design and Manufacturing. SIAM, Philadelphia, pp. 3–33.

Farouki, R.T., 1994. The conformal map $z \to z^2$ of the hodograph plane. Comput. Aided Geomet. Design 11, 363–390.

Farouki, R.T., 1996. The elastic bending energy of Pythagorean curves. Comput. Aided Geomet. Design 13, 227–241.

Farouki, R.T., 1997. Pythagorean-hodograph quintic transition curves of monotone curvature. Computer-Aided Design 29 (9), 606–610.

Farouki, R.T., Neff, C.A., 1995. Hermite interpolation by (Pythagorean-hodograph quintics. Math. Comput. 64, 1589–1609.

Farouki, R.T., Sakkalis, T., 1990. Pythagorean hodographs. IBM J. Res. Develop. 34, 726–752.
Farouki, R.T., Manni, C., Sestini, A., 2003a. Spatial C^2 PH quintic splines, Curves and Surfaces: Saint Malo 2002, in preparation.
Farouki, R.T., Han, C.Y., Manni, C., Sestini, A., 2003b. Characterization and construction of helical Pythagorean-hodograph quintic space curves, J. Comput. Appl. Math., submitted.
Farouki, R.T., Manjunathaiah, J., Nichlas, D., Yuan, G.F., Jee, S., 1998. Variable-feedrate CNC interpolators for constant material removal rates along Pythagorean-hodograph curves. Computer-Aided Design 30 (8), 631–640.
Hilgarter, E., Landsmann, G., Schicho, J., Winkler, F., 1999. Generalized offsets as envelopes of a one-parameter set of spheres, Technical Report of Res. Inst. for Symbolic Computation, Johannes Kepler Univ. Linz.
Jüttler, B., Mäuer, C., 1999. Cubic Pythagorean hodograph spline curves and applications to sweep surface modeling. Computer-Aided Design 31, 73–83.
Moon, H.P., 1999. Minkowski Pythagorean hodographs. Comput. Aided Geomet. Design 16, 739–753.
Moon, H.P., Farouki, R.T., Choi, H.I., 2001. Construction and shape analysis of PH quintic Hermite interpolants. Computer Aided Geometric Design 18, 93–115.
Peternell, M., Pottmann, H., 1997. Computing rational parametrization of canal surfaces. J. Symbolic Comput. 23, 255–266.
Walton, D.J., Meek, D.S., 1996. A Pythagorean hodograph quintic spiral. Computer-Aided Design 28 (12), 943–950.