2. Resultants

4. Let $f, g \in k[x]$ (k being a field), let the degree of f be m and the degree of g be n ($m, n > 0$). Prove:

(a) $\mathrm{Res}(f, g) = (-1)^{mn}\mathrm{Res}(g, f)$,
(b) $\mathrm{Res}(af, bg) = a^n b^m \mathrm{Res}(f, g)$, where $a, b \in k$, $a, b \neq 0$,

Do the equatities stay valid, when $m = 0$ or $n = 0$?

5. Let $f, g \in k[x]$ (k being a field), let the degree of f be m and the degree of g be n ($m, n > 0$). Prove:

(a) $\mathrm{Res}(f(x + a), g(x + a)) = \mathrm{Res}(f, g)$, where $a \in k$,
(b) $\mathrm{Res}(f(ax), g(ax)) = a^{mn}\mathrm{Res}(f, g)$, where $a \in k$, $a \neq 0$,
(c) if 0 is a root of neither f nor g, then $\mathrm{Res}(f_r, g_r) = (-1)^{mn}\mathrm{Res}(f, g)$, where $f_r(x) = x^m f(1/x)$ a $g_r(x) = x^n g(1/x)$.

6. Let $f, g \in \mathbb{C}[x]$. Prove, that the polynomial $f(x)g(y) - g(x)f(y)$ is divisible by $x - y$.

Based on the exrcise 6, for any $f, g \in \mathbb{C}[x]$ we can define so-called *Bézout matrix* $B(f, g) = (b_{ij})_{i,j=0}^{n-1}$ ($n = \max\{\deg f, \deg g\}$), where b_{ij} are the coefficients of the polynomial

$$\frac{f(x)g(y) - g(x)f(y)}{x - y} = \sum_{i,j=0}^{n-1} b_{ij} x^i y^j.$$

7. Find the Bézout matrix of the polynomials $f = 2x^2 - 5$ a $g = x^3 + 2x$.

8. Compare the determinant of the Bézout matrix $B(f, g)$ and the resultant $\mathrm{Res}(f, g)$ of the polynomials $f = f_2x^2 + f_1x + f_0$ a $g = g_1x + g_0$ ($f_2, g_1 \neq 0$).

9. Consider the cubic polynomial

$$p(x) = x^3 + ax^2 + b \in \mathbb{C}[x].$$

Find a condition for the coefficients a, b so that the polynomial p has a double root. Which one of those polynomials has a tripple root?