3. Applying resultants

10. Using resultants solve in $\mathbb{C}$:

$$\begin{aligned} x^2 + y^2 &= 10 \\ x^2 + xy + 2y^2 &= 16. \end{aligned}$$

(It is sufficient to use resultants for finding one projection of the solution set. Finish by computing the other coordinate the way you prefer.)

11. Find all $a \in \mathbb{C}$ such that the system

$$\begin{aligned} x + y &= a \\ x^2 + y^2 &= a^3 \\ x^3 + y^3 &= a^5 \end{aligned}$$

is solvable over $\mathbb{C}$.

12. Complete the proof of the lemma from the lecture:

(a) Let x appear in f (i.e. $f \notin k[y]$). Prove that for $f, g \in k[x, y]$, f being irreducible and such $f \nmid g$ in $k[x, y]$, it hold also $f \nmid g$ in $k(y)[x]$.

(b) Check that the lemma is valid also for $f \in k[y]$ (i.e. check that for an irreducible $f \in k[y]$ and for $g[x, y]$ such that $f \nmid g$, the polynomials f and g share only finitely many roots).

13. Let $\beta_1, \ldots, \beta_r \in \mathbb{R}$ be the roots of $g \in \mathbb{R}[x]$. Let $h(x, y) = g(y - x) \in \mathbb{R}[x, y]$. Find the solutions set of $h(x, y) = 0$. (It is a subset of $\mathbb{R}^2$, try to sketch it.)

14. Let $f, g \in \mathbb{Q}[x]$, let $\alpha \in \mathbb{C}$ be a root of f and $\beta \in \mathbb{C}$ be a root of g. Find a polynomial $p \in \mathbb{Q}[x]$ having $\alpha + \beta$ as one of its roots.
(Maybe you can apply the previous exercise: study the common roots of the system $f(x) = 0, g(y - x) = 0$.)

15. Find the implicit equation of the curve in the real plane, if the curve admits the rational parametrization:

$$x = \frac{t^2 - 1}{t^2 + 1}, \quad y = \frac{2t}{t^2 + 1} \quad (t \in \mathbb{R}).$$