4. The notion of algebraic variety

16. Describe the algebraic variety $V(xy, xz, yz) \subset \mathbb{A}^3(\mathbb{R})$.

17. Show that the unit circle in $\mathbb{A}^2(\mathbb{R})$ together with its center is an algebraic variety, i.e. find a polynomial equation or a system of polynomial equations such that its/their solution is exactly the given set.

18. Show that the set

$$\{(a, b) \mid a + b \in \mathbb{Z}\}$$

is not an algebraic variety v $\mathbb{A}^2(\mathbb{R})$.

19. In $\mathbb{A}^2(\mathbb{R})$ consider the sets

$$M_1 = \{(1, 1), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{3}, \tfrac{1}{3}), \ldots, (\tfrac{1}{n}, \tfrac{1}{n})\}$$
$$M_2 = \{(1, 1), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{3}, \tfrac{1}{3}), \ldots, (\tfrac{1}{n}, \tfrac{1}{n}), \ldots\}$$

For each of them find out whether it is an algebraic variety. If yes, find the defining polynomials. If not, prove.

20. Consider the set consisting of the point $(1, 1, 1)$, the point $(-1, -1, -1)$ and the points $\{(0, t, 0) \mid t \in \mathbb{C}\}$ (y-axis). Decide whether the given set is an algebraic variety in $\mathbb{A}^3(\mathbb{C})$. If yes, find the defining polynomials. If not, prove.

21. In $\mathbb{A}^3$ consider the three circles: each has its center in the beginning of the coordinates and radius 1, the first one lays in the xy-plane, the second one in the yz-plane and the third one in the xz-plane. Decide whether the three circles form an algebraic variety. If yes, find the defining polynomials. If not, prove.

22. Consider the set A of all 3×3 matrices over $\mathbb{R}$, so

$$A = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, a_{ij} \in \mathbb{R} \right\}.$$

This set can be naturally seen as the affine space $\mathbb{A}^9(\mathbb{R})$, where the coordinates are a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}.

(a) Show that the set of all singular matrices in A is an algebraic variety in this space.

(b) Show that the set of all matrices in A with the rank at most 1, is an algebraic variety in this space.

23. Consider the surface

$$V(3x^2 + 3y^2 + 3z^2 - 10xyz - 3)$$

in $\mathbb{A}^3(\mathbb{R})$. Find out whether the surface contains the line of intersection of the two planes

$$V(x - y - 2z + 2) \quad \text{and} \quad V(2x - 2y + z + 1).$$

24. (m) Let $M \subset \mathbb{A}^1(\mathbb{C})$ consists of all points of complex affine line with the real coordinate. Is M an algebraic variety in $\mathbb{A}^1(\mathbb{C})$? Give reasons for your answer.