5. Rings and ideals

25. Let R be a ring and let $G \subset R$ be a non-empty set. Prove that

$$(G) = \{r_1 g_1 + r_2 r_2 + \dots + r_k g_k\}$$

(i.e. the set of finite combinations of elements from G with coefficients from R) is an ideal in R.

26. In the ring $\mathbb{Q}[x]$ consider the ideal $I = (f, g)$ with

$$f = x^4 - x^3 - 2x^2 + 5x - 3$$
$$g = x^5 - 3x^3 + 2x^2.$$

Find a polynomial h such that $I = (h)$, and expess it as a combination of f and g over $\mathbb{Q}[x]$, i.e. find polynomials $u, v \in \mathbb{Q}[x]$ such that

$$h = u.f + v.g$$

27. Let R be a ring and let $I, J \subset R$ be ideals.

(a) Prove that IJ and $I \cap J$ are ideals.
(IJ is the set $\{a_1 b_1 + a_1 b_2 + \dots + a_n b_n \mid a_i \in I, b_i \in J\}$.)
(b) Decide, whether $IJ = I \cap J$.

28. Let R be a ring and let $I, J \subset R$ be ideals. We define $I + J := \{a + b \mid a \in I,\ b \in J\}$. Show that $I + J$ is an ideal in R. (Remark: $I + J$ is the smallest ideal containing both I and J.)

29. (m) Let R be a ring and let $I, J \subset R$ be ideals. We define $I : J := \{a \mid ab \in I\ \forall b \in J\}$.

(a) Prove that $I \subset I : J$.
(b) Prove that $I : J$ is an ideal in R.
(c) Find $I : J$ if $I = (x^3y^2, x^2y^3, y^4)$, $J = (x)$.
(d) Find $I : J$ if $I = (x^3y^2, x^2y^3, y^4)$, $J = (x^2)$.

30. In a ring R consider an increasing chain of ideals $I_0 \subset I_1 \subset I_2 \subset \dots$. Show that

$$I_\infty = \bigcup_{i=0}^{\infty} I_i$$

is an ideal in R.

31. Let R be a ring and let $I \subset R$ be an ideal. For each $m \in \mathbb{N}_0$ let

$$J_m = \{a_m \in R \mid a_m x^m + a_{m-1} x^{m-1} + \dots + a_0 \in I \text{ for some } a_0, \dots, a_{m-1} \in R\},$$

so J_m is the set containing the leading coefficients of polynomials in I of degree m, together with 0.

(a) Prove that J_m is an ideal in R.
(b) Prove that $J_m \subset J_{m+1}$ for all $m \in \mathbb{N}_0$.