6. Ideals and varieties, Hilbert's basis theorem

32. Given two systems of linear equations decide whether they describe the same linear variety in $\mathbb{A}^3(\mathbb{R})$, i.e. find out whether $V(f_1, f_2(, f_3)) = V(g_1, g_2)$:

(a) $f_1 = x + y + z - 1, f_2 = x - y + 2z - 4,$
$g_1 = x + 5y - z + 5, g_2 = 3x + y + z - 2.$

(b) $f_1 = 2x + 3y - z, f_2 = x + y - 1, f_3 = x + z - 3,$
$g_1 = x + 3y - 2z + 3, g_2 = y - z + 2.$

(c) Given are two linear vatieries $X_1 = V(f_1, \dots, f_r)$, $X_2 = V(g_1, \dots, g_s) \subset \mathbb{A}^n$ ($f_i, g_j \in k[x_1, \dots, x_n]$ are linear). Try to design an algorithm deciding whehther $X_1 = X_2$.

33. There are more ways for describing the two-point-set from the lecture

$$X = \{(1, 2), (3, 4)\} \subset \mathbb{A}^2(\mathbb{Q})$$

as the solution set of a system of polynomial equations. Show that

$$((x - 2)^2 + (y - 3)^2 - 2, x - y + 1) = ((x - 1)(x - 3), x - y + 1).$$

In other words, show that the two sets of polynomials generate the same ideal. (Actually, this claim is stronger than the claim that they describe the same algebraic variety.)

34. (m) Let the field k be infinite and let $X \subset \mathbb{A}^n(k)$ be a finite set. We know already that X is an affine variety. Show that X is the zero set of n polynomials, i.e. that there exist $f_1, \dots, f_n \in k[x_1, \dots, x_n]$ such that $X = V(f_1, \dots, f_n)$. (Hint: interpolation.)

35. Let $I, J \subset k[x_1, \dots x_n]$ be ideals and let $X = V(I)$, $Y = V(J)$ be corresponding algebraic varieties. What can you say about the varieties $V(IJ)$, $V(I \cap J)$, $V(I + J)$?

36. (m) Let k be a field. Is the ring $k[x_1, x_2, \dots]$ of polynomials in infinitely many variables noetherian?

37. (m) Consider the set of real functions that are continuous on the interval $[0, 1] \subset \mathbb{R}$. This set is a ring. Show that this ring is not noetherian.

38. Let $S \subset \mathbb{A}^n(k)$ and let $I(S) \subset k[x_1, \dots, x_n]$ (a set defined in the lecture). Show that $I(S)$ is an ideal in $k[x_1, \dots, x_n]$.

39. Prove the proposition 4.15 from the lecture. (You prove (i) and (ii) by just rewriting the definitions. The claim (iii) will follow from (i) and (ii).)