7. The radical of an ideal, Nullstellensatz

40. For an ideal I in a ring R show that $\sqrt{I}$ is an ideal.

41. Let I be an ideal in a ring R. Which of the following statements are true?

(a) If I is maximal, then it is radical.
(b) If I is a prime ideal, then it is radical.
(c) If I is radical, then it is maximal.
(d) If I is radical, then it is a prime ideal.

Give reasons for your answers.

42. Let $I, J \subset k[x_1, \ldots, x_n]$ be ideals. Show that

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

43. Let $X \subset \mathbb{A}^n$. Show that $I(X)$ is a radical ideal.

44. Let $I = (x^2y + xy^2, x^2y - xy^2) \subset k[x, y]$ be an ideal. Find $\sqrt{I}$.

45. Let $I = (x^4 - 2x^2y^2 + y^2, x^4 - y^2) \subset \mathbb{C}[x, y]$. Find out whether $x^2 - y^2 \in \sqrt{I}$ and whether $x^2 + y^2 \in \sqrt{I}$.

46. Find the radical of $(x^2y^4 - y^2, 1 - x^3y^3) \subset \mathbb{C}[x, y]$.

47. Let k be a field that is not algebraically closed and let $X \subset \mathbb{A}^n(k)$ be an algebraic variety. Show that there is a polynomial $g \in k[x_1, \ldots, x_n]$ such that $X = V(g)$.
(Hint: is there a polynomial $h \in k[y_1, \ldots, y_r]$ such that $V(h) = \{(0, \ldots, 0)\}$? If yes, can you make use of it?)