9. Coordinate ring of variety, morphisms.

54. Let φ be a ring homomorphism (rings are commutative with 1)

$$\varphi\colon\ R_1 \to R_2.$$

The *kernel* of the homomorphism φ is the set

$$\ker\varphi = \{r \in R_1 \mid \varphi(r) = 0\}.$$

Show that $\ker\varphi$ is an ideal in the ring R_1.

55. Let φ be a ring homomorphism (rings are commutative with 1) and let J be an ideal in R_2. Show that the preimage of the ideal J in φ, i.e. the set

$$\varphi^{-1}(J) = \{r \in R_1 \mid \varphi(r) \in J\}$$

is an ideal in R_1.

56. Let $X = V(x^2 + y^2 - 1, x - 1) \subset \mathbb{A}^2(\mathbb{C})$. Find $\mathbb{C}[X]$.

57. Let $X = V(x^2 + y^2 - 1, y) \subset \mathbb{A}^2(\mathbb{C})$. Find $\mathbb{C}[X]$.

58. Consider the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ given by $\sigma:\ (x, y) \mapsto (x, xy)$.

(a) What is the image of $\mathbb{A}^2$ under the morphism? With respect to Zariski topology, is the image of $\mathbb{A}^2$ a closed set? Is it open?
(b) Find the image of the line $V(y - x)$ under this morphism.
(c) Try to find the image of the unit circle under this morphism.

(The given morphism is of particular interest in algebraic geometry. It is used for studying sigularities of plane algebraic curves.)

In exercises 56 a 57 you should describe the structure of the considered sets. It is not enough just to write the definition $\mathbb{C}[X] = \mathbb{C}[x, y]/I(X)$ and plug in the correct ideal for $I(X)$.