10. Morphisms and rational maps.

59. Let X and Y be isomorphic algebraic varieties. Show that if X is irreducible, then also Y is irreducible.

60. Using the previous exercise show that the twisted cubic $V(y - x^2, z - x^3) \subset \mathbb{A}^3(\mathbb{C})$ is irreducible.

61. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a regular matrix over an infinite field k. Show that the rational map

$$f : \mathbb{A}^1(k) \to \mathbb{A}^1(k), \quad x \mapsto \frac{ax + b}{cx + d}$$

is dominant. Is f a birational equivalence?

62. Let $X = V(y^2 - xz, z^2 - y^3) \subset \mathbb{A}^3(\mathbb{C})$. Decompose X into irreducible components. Prove that each component is rational (i.e. birational to $\mathbb{A}^n(\mathbb{C})$ for some n).

63. Let C be the cubic from the previous example, $C = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2(\mathbb{C})$. Show that the function y/x is not regular, i.e. $y/x \notin \mathbb{C}[C]$. In which points of C is the function regular?

64. The cubic curve $C = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2(\mathbb{C})$ is singular in $(0, 0)$. Sketch the curve (its real points). Find a birational equivalence of the curve with an affine line. (Hint: you can proceed like in the case of stereograhic projection of a circle. In this case, consider lines throug the singular point and project the curve onto the line $x = 1$.)

65. Let $X \subset \mathbb{A}^n$ be the hypersurface defined by the polynomial

$$F_{n-1}(x_1, \dots, x_n) + F_n(x_1, \dots, x_n),$$

with F_{n-1} resp. F_n being homogenious of degree $n - 1$ resp. n. Prove, that if X is irreducible, then it is birational to $\mathbb{A}^{n-1}$.