12. Buchberger's algorithm.

70. Using Bucherger's algorithm, find a Gröbner basis of the ideal $(z - x^5, y - x^3) \subset k[x, y, z]$. Use

(a) lexicographic ordering $(x > y > z)$,
(b) graded reverse lexicographic ordering.

Give also computations and reductions of S-polynomials.

71. Buchberger's algorithm is a generalization of more special cases which you already know. Try to recognize them:

(a) Let $I = (f, g) \subset k[x]$ be an ideal in a polynomial ring with one variable. What can you say about Buchberger's algorithm in this case? (Hint: you might try an example, e.g. for $I = (x^3 - 1, x^5 - 1) \subset \mathbb{R}[x]$, check (using Buchberger's algorithm) whether $x^3 + x^2 - 2$ belongs to I.)
(b) Let $I \subset k[x_1, \ldots, x_n]$ be an ideal generated by linear polynomials (and so defining a linear variety). What can you say about Buchberger's algorithm?

72. Let

$$g_1 = y^3 + 3y^2z^3, \quad g_2 = xyz + 3xz^4, \quad g_3 = 4x^2z - 7y^2$$

be polynomials in $k[x, y, z]$ and let $I = (g_1, g_2, g_3)$ be an ideal. Consider the lexicographic ordering of monomials with $x > y > z$. Is it possible to find $g \in I$ such that

$$\mathrm{LT}(g) \notin (\mathrm{LT}(g_1), \mathrm{LT}(g_2), \mathrm{LT}(g_3))?$$

If yes, find it. If no, prove.

73. Using Gröbner bases, solve:

$$\begin{aligned} x^2 + y^2 + z^2 &= 4, \\ x^2 + 2y^2 &= 5, \\ xz &= 1. \end{aligned}$$

Help yourself with a computer algebra system:

- for example, Singular (`https://www.singular.uni-kl.de:8003/`),
- or Sage (`https://sagecell.sagemath.org/`),
- or any other that you prefer.